1. Introduction
2. Cartesian Vectors and Tensors: Their Algebra
3. Cartesian Vectors and Tensors: Their Calculus
4. The Kinematics of Fluid Motion
5. Stress in Fluids
6. Equations of Motion and Energy in Cartesian Coordinates
7. Solution of the Partial Differential Equations
8. Laminar Flows with Dependence on One Dimension
9. Multidimensional Laminar Flow
10. Flow with Free Surface
11. Numerical Simulation

Introduction

Topics Covered in this Chapter

- Prerequisites and text books
- Scalar, vector and tensor fields
- Curves, surfaces, and volumes
- Coordinate systems
- Units
- Continuum approximation
- Densities, potential gradients, and fluxes
- Velocity: a measure of flux by convection
- Density
- Species concentration
- Energy (heat)
- Porous media
- Momentum
- Electricity and Magnetism

Introduction

This course is designed as a first level graduate course in transport phenomena. Undergraduate courses generally start with simple example problems and lead to more complex problems. With this approach, the student must learn the fundamental principles by induction. The approach used here is to teach the fundamental principles and then deduce the analysis for example problems. The example problems are classical problems that should be familiar to all Ph.D. Chemical Engineering graduates. These problems will be presented not only as an exercise with analytical or numerical solutions but also as simulated experiments which are to be interpreted and graphically displayed for presentation.

Prerequisites and text books

Students in this class are expected to have a background corresponding to a BS degree in Chemical Engineering. This includes a course in multivariable calculus, which covers the algebra and calculus of vectors fields on volumes,

surfaces, and curves of 3-D space and time. Courses in ordinary and partial differential equations are a prerequisite. Some elementary understanding of fluid mechanics is expected from a course in transport phenomena, fluid mechanics, or physics. It is assumed that not all students have the prerequisite background. Thus, material such as vector algebra and calculus will be briefly reviewed and exercise problems assigned that will require more reading from the student if they are not already familiar with the material.

The two required textbooks for this course are R. Aris, *Vectors, Tensors, andthe Basic Equations of Fluid Mechanics* and Bird, Stewart, and Lightfoot, *Transport Phenomena*. Several of the classical problems are from S. W. Churchill, *Viscous Flows, The Practical Use of Theory*. The classical textbook, Feyman, Leighton, and Sands, *The Feyman Lectures on Physics, Volume II* is highly recommended for its clarity of presentation of vector fields and physical phenomena. The students are expected to be competent in MATLAB, FORTRAN, and EXCEL and have access to *Numerical Recipes in FORTRAN*.

The following table is a suggested book list for independent studies in transport phenomena.

Author	**Title**	**Publisher**	**Year**
L.D. Landau andE. M. Lifshitz	Fluid Mechanics, 2nd Ed.	Butterworth	1987
V. G. Levich	Physicochemical Hydrodynamics	Prentice-Hall	1962
S. Chandrasekhar	Hydrodynamics and Hydromagnetic Stability	Dover	1961

Author	Title	Publisher	Year
H. Schlichting	Boundary Layer Theory	McGraw-Hill	1960
H. Lamb	Hydrodynamics	Dover	1932
S. Goldstein	Modern Developments in Fluid Dynamics	Dover	1965
W. E. Langlois	Slow Viscous Flow	Macmillan	1964
J. Happel, H. Brenner	Low Reynolds Number Hydrodynamics	Kluwer	1973
G. K. Batchelor	An Introduction to Fluid Mechanics	Cambridge	1967
S.-I. Pai	Viscous Flow Theory I Laminar Flow	Van Nostrand	1956
M. Van Dyke	Perturbation Methods in Fluid Mechanics	Academic Press	1964
S. W. Churchill	Inertial Flows	Etaner	1980
S. W. Churchill	Viscous Flows	Butterworths	1988
R. F. Probstein	Physicochemical Hydrodynamics	Butterworth-Heinemann	1989

Author	Title	Publisher	Year
S. Middleman	An Introduction to Fluid Dynamics	John Wiley	1998
S. Middleman	An Introduction to Mass and Heat Transfer	John Wiley	1998
E. L. Koschmeider	Benard Cells and Taylor Vortices	Cambridge	1993
W.-J. Yang	Handbook of Flow Visualization	Taylor & Francis	1989
W.-J. Yang	Computer-Assisted Flow Visualization	CRC Press	1994
A. J. Chorin	Computational Fluid Mechanics	Academic Press	1989
A. J. Chorin, J. E. Marsden	A Mathematical Introduction to Fluid Mechanics	Springer-Verlag	1993
L. C. Wrobel,C. A. Brebbia	Computational Modeling of Free and Moving Boundary Problems, Vol. 1 Fluid Flow	Computational Mechanics Publications	1991
M. J. Baines,K. W. Morton	Numerical Methods for Fluid Dynamics	Oxford	1993

Author	Title	Publisher	Year
W. E. Schiesser,C. A. Silebi	Computational Transport Phenomena	Cambridge	1997
N. Ida, J. P. A. Bastos	Electro-Magnetics and Calculation of Fields	Springer	1997
L. G. Leal	Laminar Flow and Convective Transport Processes	Butterworth	1992
W. M. Deen	Analysis of Transport Phenomena	Oxford	1998
C. S. Jog	Foundations and Applications of Mechanics, Vol. I, Continuum Mechanics	CRC Press	2002
C. S. Jog	Foundations and Applications of Mechanics, Vol. II, Fluid Mechanics	CRC Press	2002
T.J. Chung	Computational Fluid Dynamics	Cambridge	2002
R. J. Kee, M.E. Coltrin, P. Glarborg	Chemically Reacting Flow	Wiley - Interscience	2003

Author	Title	Publisher	Year
Z.U.A. Warsi	Fluid Dynamics; Theoretical and Computational Approaches	Taylor & Francis	2006
Y. A. Cengel and J. M. Cimbala	Fluid Mechanics; Fundamentals and Applications	McGraw Hill	2006

Transport phenomena book list

Scalar, vector and tensor fields

Scalars, vectors, and matrices are concepts that may have been introduced to the student in a course in linear algebra. Here, scalar, vector, and tensor fields are entities that are defined over some region of 3-D space and time. It is implicit that they are a function of the spatial coordinates and time, i.e., $\varphi = \varphi(x, y, z, t) = \varphi(x, t)$. The spatial coordinates are expressed as Cartesian coordinates in this class. However, vectors and tensors are physical entities that are independent of the choice of spatial coordinates even though their components depend on the choice of coordinates.

Scalar fields have a single number, a scalar, at each point in space. An example is the temperature of a body. The temperature field is usually expressed visually by a contour map showing curves of constant temperature or isotherms. An alternative visual display of a scalar field is a color map with the value of the scalar scaled to a gray scale, hue, or saturation. The values of the scalar field are continuous with the exception of definable surfaces of discontinuity. An example is the density of two fluids separated by an interface. Media that are chaotic and discontinuous on a microscopic scale may be described by an average value in a *representative elementary volume* that is large compared to the microscopic heterogeneity but small compared to macroscopic variations. An example is the porosity of a porous solid.

Vector fields have a magnitude and direction associated with each point in space. An example is the velocity field of a fluid in motion. Vector fields in two dimensions can be visually expressed as field lines that are everywhere tangent to the vector field and whose separation quantifies the magnitude of the field. Streamlines are the field lines of the velocity field. Alternatively, a vector field in two dimensions can be visually expressed by arrows whose directions are parallel to the vector and having a width and/or length that scales to the magnitude of the vector. These graphical representations of vector fields are not useful in three dimensions. In general, a vector field in 3-D can be expressed in terms of its components projected on to the axis of a coordinate system. Thus, a vector field may have different components when projected on to different coordinate systems. Since a vector is a physical entity, the components in different frames of reference transform by prescribed rules. The position of a point in space relative to an origin is a vector defined by the distance and direction. Special vectors having a magnitude of unity are called unit vectors and are used to define a direction such as coordinate directions or the normal direction to a surface. We will denote vectors with bold face letters, e.g., **v, x,** or **n**.

Tensors are physical entities associated with two directions. For example, the stress tensor represents the force per unit area, each of which are directional quantities. The velocity gradient is a tensor. Transport coefficients, such as the thermal conductivity, are tensors, which transform a potential gradient to a flux, each of which are vectors. The components of a tensor in a particular coordinate system are represented by a 3×3 matrix. Since the tensor is a physical entity that is independent of the coordinate system, the components must satisfy certain transformation rules between coordinate systems. In particular, a set of three directions called the principal directions can be found to transform the components of the tensor to a diagonal matrix. This corresponds to finding the eigenvectors of a matrix and the components correspond to the eigenvalues. Bold face letters will also denote tensors. The stress tensor will be denoted by **T** or **τ**, depending on whether discussing Aris or BSL, respectively.

Curves, surfaces, and volumes

We will be dealing with regions of space, *V*, having volume that may be bound by surfaces, *S*, having area. Regions of the surface may be bound by a closed curve, *C*, having length.

Surfaces are defined by one relationship between the spatial coordinates.
Equation:

$$x^3 = f\left(x^1, x^2\right),\ or\ F\left(x^1, x^2, x^3\right) = 0,\ or\ F\left(x\right) = 0$$

Alternatively, a pair of surface coordinates, u^1, u^2 can define a surface.
Equation:

$$x^i = x^i\left(u^1, u^2\right),\ i = 1, 2, 3\ or\ \mathbf{x} = \mathbf{x}\left(u^1, u^2\right)$$

Each point on the surface that has continuous first derivatives has associated with it the normal vector, **n**, a unit vector that is perpendicular or normal to the surface and is outwardly directed if it is a closed surface. Fluid-fluid interfaces need to also be characterized by the mean curvature, H, at each point on the surface to describe the normal component of the momentum balance across the interface. The flux of a vector, **f**, across a differential element of the surface is denoted as follows, i.e. the normal component of the flux vector multiplied by the differential area.
Equation:

$$\mathbf{f} \cdot \mathbf{n}\ da$$

Curves are defined by two relationships between the spatial coordinates or by the intersection of two surfaces.
Equation:

$$f_1\left(x^1, x^2, x^3\right) = 0\ and\ f_2\left(x^1, x^2, x^3\right) = 0,\ or\ f_1\left(x\right) = 0\ and\ f_2\left(x\right) = 0$$

Alternatively, a curve in space can be parameterized by a single parameter, such as the distance along the curve, *s* or time, *t*.
Equation:

$$\mathbf{x} = \mathbf{x}(s)$$

The tangent vector is a unit vector that is tangent to each point on the curve.
Equation:

$$\boldsymbol{\tau} = d\mathbf{x}(s)/ds$$

The component of a vector, **f**, tangent to a differential element of a curve is denoted as follows.
Equation:

$$\mathbf{f} \cdot \boldsymbol{\tau}\ ds$$

If the parameter along the curve is time, the differential of position with respect to time is the velocity vector and the differential of velocity is acceleration.
Equation:

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

Coordinate systems

Scalars, vectors, and tensors are physical entities that are independent of the choice of coordinate systems. However, the components of vectors and tensors depend on the choice of coordinate systems. The algebra and calculus of vectors and tensors will be illustrated here with Cartesian coordinate systems but these operations are valid with any coordinate system. The student is suggested to read Aris to learn about curvilinear coordinate systems. Bird, Stewart, and Lightfoot express the components of the relevant vector and tensor equations in Cartesian, cylindrical polar, and spherical polar coordinate systems.

Cartesian coordinates have coordinate axes that have the same direction in the entire space and the coordinate values have the units of length. Curvilinear coordinates, in general, may have coordinate axis that are in different directions at different locations in space and have coordinate values that may not have the units of length, e.g., θ in the cylindrical polar system. If (y^1, y^2, y^3) are Cartesian coordinates and (x^1, x^2, x^3) are curvilinear coordinates, a differential length is related to the differential of the coordinates by the following relations.

$$
\begin{aligned}
ds^2 &= \sum_{k=1}^{3} dy^k dy^k \\
dy^k &= \tfrac{\partial y^k}{\partial x^i} dx^i \equiv \tfrac{\partial y^k}{\partial x^1} dx^1 + \tfrac{\partial y^k}{\partial x^2} dx^2 + \tfrac{\partial y^k}{\partial x^3} dx^3 \\
ds^2 &= \sum_{k=1}^{3} \left(\frac{\partial y^k}{\partial x^i} dx^i \right) \left(\frac{\partial y^k}{\partial x^j} dx^j \right) \\
&\qquad\qquad = g_{ij} dx_i dx_j \\
g_{ij} &= \sum_{k=1}^{3} \left(\frac{\partial y^k}{\partial x^i} \right) \left(\frac{\partial y^k}{\partial x^j} \right)
\end{aligned}
$$

where g_{ij} are components of the metric tensor which transforms differential of the coordinates to differential of length. Summation is understood for repeated indices. Calculus in a curvilinear coordinate system will require the metric tensor.

A differential element of volume in curvilinear coordinate system is related to differentials of the coordinates by the square root of the determinate of the metric tensor or the Jacobian, *J*.

Equation:

$$
\begin{aligned}
dV &= dy^1\, dy^2\, dy^3 \\
&= \varepsilon_{ijk} \tfrac{\partial y^i}{\partial x^1} \tfrac{\partial y^j}{\partial x^2} \tfrac{\partial y^k}{\partial x^3}\, dx^1\, dx^2\, dx^3 \\
&= g^{1/2}\, dx^1\, dx^2\, dx^3 \\
&= J\, dx^1\, dx^2\, dx^3
\end{aligned}
$$

Henceforth, Cartesian coordinates with subscript notation will be used.

Units

Dimensional quantities will be used in equations without explicit specification of units because it is understood that they will have the SI system of units. The SI units and mks units are similar with some exceptions as in electricity and magnetism. The following table lists the SI units of the quantities used in this course and the conversion factor needed to convert the quantity from some customary units to SI units. Multiply the quantity in customary units by the conversion factor to obtain the quantity in SI units. The following is taken from, *The SI Metric System of Units and SPE METRIC STANDARD*, Society of Petroleum Engineers.

Quantity	**SI unit**	**Customary unit**	**Conversion factor**
Length	m	ft	3.048 E-01
Mass	kg	lbm	4.535 924 E-01
Time	s	s	1.0
Temperature	°K	°R	5/9
Pressure, stress	Pa	psi	6.894 757 E+03
Density	kg/m^3	g/cm^3	1.0 E+03

Quantity	SI unit	Customary unit	Conversion factor
Force	N	lbf	4.448 222 E+00
Flow rate	m^3/s	U.S. gal/min	6.309 020 E-05
Diffusivity	m^2/s	cm^2/s	1.0 E-04
Thermal conductivity	W/(m•K)	Btu/(hr-ft^2-°F/ft)	1.730 735 E+00
Heat transfer coefficient	W/(m^2•K)	Btu/(hr-ft^2-°F)	5.678 263 E-06
Permeability	m^2	darcy	9.869 233 E-13
Surface tension	N/m	dyne/cm	1.0 E-03
Viscosity (dynamic)	Pa•s	cp	1.0 E-03

SI units and conversion factors

Continuum approximation

The calculus of scalar, vector, and tensor fields require that these quantities be piecewise continuous down to infinitesimal dimensions. However, quantities such as density, pressure, and velocity become ambiguous or stochastic at the scale of molecular dimensions. Thus the fields discussed here are the average value of the quantity over a *representative elementary volume*, *REV*, of space that is large compared to molecular dimensions but small compared to the macroscopic variation of the quantities. The size of the REV depends on the

scale that a problem is being investigated. For example, suppose one is investigating a fixed bed catalyst reactor. As a first order approximation for design purposes, the reactor may be modeled as a one-dimensional system with the cross-section of the reactor approximated as the REV. However, is one is investigating instabilities and channeling, the bed may be modeled in 2-D with the REV being a volume that is small compared to the macroscopic dimensions of the reactor but large compared to the size of the catalyst particles. If one is optimizing the transport-limited kinetics of the reactor, then the REV may be small compared to the size of the catalyst particle. If one is optimizing the balance between transport-limited and surface reaction rate limited kinetics, the REV may be small enough to describe the surface morphology of the catalyst particle. However, the molecular dynamics of the surface reaction is beyond the realm of transport phenomena.

Densities, potential gradients, and fluxes

Velocity: and flux by convection. Transport or flux of the various quantities discussed in this course will be due to convection (or advection) or due to the gradient of a potential. Common to all of these transport process is the convective transport resulting from the net or average motion of the molecules or the velocity field, **v**. The convective flux of a quantity is equal to the product of the density of that quantity and the velocity. In this sense, the velocity vector can be interpreted as a "volumetric flux" as it has the units of the flow of volume across a unit area of surface per unit of time. Because the flux by convection is common to all forms of transport, the integral and differential calculus that follow the convective motion of the fluid will be defined. These will be known as the Reynolds' transport theorem and the convective or material derivative.

Mass density and mass flux. If ρ is the mass density, the mass flux is $\rho\mathbf{v}$.

Species concentration. Suppose the concentration of species A in a mixture is denoted by C_A. The convective flux of species A is $C_A\,\mathbf{v}$. Fick's law of diffusion gives the diffusive flux of A.
Equation:

$$\mathbf{J}_A = -\mathbf{D}_A \bullet \nabla C_A$$

The diffusivity, $\mathbf{D}_A$, is in general a tensor but in an isotropic medium, it is usually expressed as a scalar.

<u>Internal energy (heat).</u> The density of internal energy is the product of density and specific internal energy, ρE. The convective flux is ρE **v**. For an incompressible fluid, the convective flux becomes $\rho\ Cp\ (T\text{-}To)$ **v**. The conductive heat flux, **q**, is given by Fourier's law for conduction of heat,
Equation:

$$\mathbf{q} = -\mathbf{k} \bullet \nabla T$$

where **k** is the thermal conductivity tensor (note: same symbol as for permeability).

<u>Porous media</u>. The density of a single fluid phase per unit bulk volume of porous media is $\phi\rho$, where ϕ is the porosity. Darcy's law gives the volumetric flux, superficial velocity, or Darcy's velocity as a function of a potential gradient.
Equation:

$$\begin{aligned}\mathbf{u} &= -\frac{\mathbf{k}}{\mu} \bullet (\nabla p - \rho\,\mathbf{g}) \\ &= \phi\,\mathbf{v}\end{aligned}$$

where **k** is the permeability tensor and **v** is the interstitial velocity or the average velocity of the fluid in the pore space. Darcy's law is the momentum balance for a fluid in porous media at low Reynolds number.

<u>Momentum balance</u>. Newton's law of motion for an element of fluid is described by Cauchy's equation of motion.
Equation:

$$\begin{aligned}\rho\,\mathbf{a} &= \rho\,\frac{d\mathbf{v}}{dt} \\ &= \rho\,\mathbf{f} + \nabla \bullet \mathbf{T}\end{aligned}$$

where **f** is the sum of body forces and **T** is the stress tensor. The stress tensor can be interpreted as the flux of force acting on the bounding surface of an element of fluid.

Equation:

$$\begin{aligned}\iiint\limits_{V(t)} (\rho\,\mathbf{a} - \rho\,\mathbf{f})\,dV &= \iiint\limits_{V(t)} \nabla \bullet \mathbf{T}\,dV \\ &= \iint\limits_{S(t)} \mathbf{T} \bullet \mathbf{n}\,dS\end{aligned}$$

The stress tensor for a Newtonian fluid is as follows.

Equation:

$$\begin{aligned}&\mathbf{T} = (-p + \lambda\,\Theta)\,\mathbf{I} + 2\mu\,\mathbf{e} \\ &\mathbf{e} = \tfrac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^t\right)\end{aligned}$$

where *p* is the thermodynamic pressure, Θ is the divergence of velocity, μ is the coefficient of shear viscosity, (λ+2/3μ) coefficient of bulk viscosity, and **e** is the rate of deformation tensor. Thus the anisotropic part (not identical in all directions) of the stress tensor is proportional to the symmetric part of the velocity gradient tensor and the constant of proportionality is the shear viscosity.

<u>Electricity and Magnetism</u>. We will not be solving problems in electricity and magnetism but the fundamental equations are presented here to illustrate the similarity between the field theory of transport phenomena and the classical field theory of electricity and magnetism. The Maxwell's equations and the constitutive equations are as follows.

Equation:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$
$$\nabla \bullet \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \bullet \mathbf{D} = \rho$$
$$\text{Constitutive equations:}$$
$$\mathbf{B} = \mu \mathbf{H}$$
$$\mathbf{D} = \varepsilon \mathbf{E}$$
$$\mathbf{J} = \sigma \mathbf{E}$$

where

- **E** electric field intensity
- **D** electric flux density or electric induction
- **H** magnetic field intensity
- **B** magnetic flux density or magnetic induction
- **J** electric current density
- **ρ** charge density
- **μ** magnetic permeability (tensor if anisotropic)
- **ε**electric permittivity (tensor if anisotropic)
- **σ**electric conductivity (tensor if anisotropic)

When the fields are quasi-static, the coupling between the electric and magnetic fields simplify and the fields can be represented by potentials.
Equation:

$$\mathbf{E} = -\nabla V$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

where *V* is the electric potential and **A** is the vector potential. The electric potential is analogous to the flow potential for invicid, irrotational flow and the vector potential is analogous to the stream function in two-dimensional, incompressible flow.

Reading assignment

Note: Read Chapter 1 and Appendix A and B of Aris.

Cartesian Vectors and Tensors: Their Algebra

Topics Covered in this Chapter

- Definition of a vector
- Examples of vectors
- Scalar multiplication
- Addition of vectors – coplanar vectors
- Unit vectors
- A basis of non-coplanar vectors
- Scalar product – orthogonality
- Directional cosines for coordinate transformation
- Vector product
- Velocity due to rigid body rotations
- Triple scalar product
- Triple vector product
- Second order tensors
- Examples of second order tensors
- Scalar multiplication and addition
- Contraction and multiplication
- The vector of an antisymmetric tensor
- Canonical form of a symmetric tensor

Note: Reading Assignment: Chapter 2 of Aris, Appendix A of BSL

The algebra of vectors and tensors will be described here with Cartesian coordinates so the student can see the operations in terms of its components without the complexity of curvilinear coordinate systems.

Definition of a vector

Suppose x_i , i.e., (x_1, x_2, x_3), are the Cartesian coordinates of a point P in a frame of reference, 0123. Let 0123 be another Cartesian frame of reference with the same origin but defined by a rigid rotation. The

coordinates of the point P in the new frame of reference is x_j where the coordinates are related to those in the old frame as follows.
Equation:

$$x_j = l_{ij} \;\; x_i = l_{1j}x_1 + l_{2j}x_2 + l_{3j}x_3$$
$$x_i = l_{ij} \;\; x_j = l_{i1}x_1 + l_{i2}x_2 + l_{i3}x_3$$

where l_{ij} are the cosine of the angle between the old and new coordinate systems. Summation over repeated indices is understood when a term or a product appears with a common index.

Cartesian Vector

A Cartesian vector, a, in three dimensions is a quantity with three components a_1, a_2, a_3 in the frame of reference 0123, which, under rotation of the coordinate frame to 0123 , become components a_1, a_2, a_3 , where

Equation:

$$a_j = l_{ij}a_i$$

Examples of vectors

In Cartesian coordinates, the length of the position vector of a point from the origin is equal to the square root of the sum of the square of the coordinates. The *magnitude* of a vector, **a**, is defined as follows.
Equation:

$$|\mathbf{a}| = (a_i a_i)^{1/2}$$

A vector with a magnitude of unity is called a *unit vector*. The vector, **a**/|**a**|, is a unit vector with the direction of **a**. Its components are equal to the cosine of the angle between **a** and the coordinate axis. Some special unit

vectors are the unit vectors in the direction of the coordinate axis and the normal vector of a surface.

Scalar multiplication

If α is a scalar and **a** is a vector, the product α**a** is a vector with components, α_{ai}, magnitude α|**a**|, and the same direction as **a**.

Addition of vectors – Coplanar vectors

If **a** and **b** are vectors with components a_i and b_i, then the sum of **a** and **b** is a vector with components, a_i+b_i.

The order and association of the addition of vectors are immaterial.
Equation:

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$$
$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$$

The subtraction of one vector from another is the same as multiplying one by the scalar (-1) and adding the resulting vectors.

If **a** and **b** are two vectors from the same origin, they are *colinear* or parallel if one is a linear combination of the other, i.e., they both have the same direction. If **a** and **b** are two vectors from the same origin, then all linear combination of **a** and **b** are in the same plane as **a** and **b**, i.,e., they are *coplanar*. We will prove this statement when we get to the triple scalar product.

Unit vectors

The unit vectors in the direction of a set of mutually orthogonal coordinate axis are defined as follows.

Equation:

$$\mathbf{e}_{(1)} = \begin{matrix} 1 \\ 0 \\ 0 \end{matrix}, \mathbf{e}_{(2)} = \begin{matrix} 0 \\ 1 \\ 0 \end{matrix}, \mathbf{e}_{(3)} = \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$$

The suffixes to **e** are enclosed in parentheses to show that they do not denote components. A vector, **a**, can be expressed in terms of its components, (a_1, a_2,a_3) and the unit vectors.
Equation:

$$\mathbf{a} = a_1\mathbf{e}_{(1)} + a_2\mathbf{e}_{(2)} + a_3\mathbf{e}_{(3)}$$

This equation can be multiplied and divided by the magnitude of **a** to express the vector in terms of its magnitude and direction.
Equation:

$$\begin{aligned} \mathbf{a} &= |\mathbf{a}| \left(\tfrac{a_1}{|\mathbf{a}|}\mathbf{e}_{(1)} + \tfrac{a_2}{|\mathbf{a}|}\mathbf{e}_{(2)} + \tfrac{a_3}{|\mathbf{a}|}\mathbf{e}_{(3)} \right) \\ &= |\mathbf{a}| \left(\lambda_1\mathbf{e}_{(1)} + \lambda_2\mathbf{e}_{(2)} + \lambda_3\mathbf{e}_{(3)} \right) \end{aligned}$$

where λ_i are the directional cosines of **a**.

A special unit vector we will use often is the normal vector to a surface, **n**. The components of the normal vector are the directional cosines of the normal direction to the surface.

Scalar product – Orthogonality

The *scalar product* (or *dot product*) of two vectors, **a** and **b** is defined as
Equation:

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$$

where θ is the angle between the two vectors. If the two vectors are perpendicular to each other, i.e., they are *orthogonal*, then the scalar product is zero. The unit vectors along the Cartesian coordinate axis are orthogonal and their scalar product is equal to the Kronecker delta.
Equation:

$$\begin{aligned}\mathbf{e}_{(i)} \bullet \mathbf{e}_{(j)} &= \delta_{ij} \\ &= \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}\end{aligned}$$

The scalar product is commutative and distributive. The cosine of the angle measured from **a** to **b** is the same as measured from **b** to **a**. Thus the scalar product can be expressed in terms of the components of the vectors.
Equation:

$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} &= \left(a_1\mathbf{e}_{(1)} + a_2\mathbf{e}_{(2)} + a_3\mathbf{e}_{(3)}\right) \bullet \left(b_1\mathbf{e}_{(1)} + b_2\mathbf{e}_{(2)} + b_3\mathbf{e}_{(3)}\right) \\ &= a_i b_j \delta_{ij} \\ &= a_i b_i\end{aligned}$$

The scalar product of a vector with itself is the square of the magnitude of the vector.
Equation:

$$\begin{aligned}\mathbf{a} \bullet \mathbf{a} &= |\mathbf{a}|\,|\mathbf{a}| \cos 0 \\ &= |\mathbf{a}|^2 \\ \mathbf{a} \bullet \mathbf{a} &= a_i a_i \\ &= |\mathbf{a}|^2\end{aligned}$$

The most common application of the scalar product is the projection or component of a vector in the direction of another vector. For example, suppose **n** is a unit vector (e.g., the normal to a surface) the component of **a** in the direction of **n** is as follows.
Equation:

$$\mathbf{a} \bullet \mathbf{n} = |\mathbf{a}| \cos\theta$$

Directional Cosines for Coordinate Transformation

The properties of the directional cosines for the rotation of the Cartesian coordinate reference frame can now be easily illustrated. Suppose the unit vectors in the original system is $\mathbf{e}_{(i)}$ and in the rotated system is $\overline{\mathbf{e}}_{(j)}$. The components of the unit vector, $\overline{\mathbf{e}}_{(j)}$, in the original reference frame is l_{ij}. This can be expressed as the scalar product.

Equation:

$$\begin{aligned}
&\mathbf{e_{(j)}} = l_{1j}\mathbf{e_{(1)}} + l_{2j}\mathbf{e_{(2)}} + l_{3j}\mathbf{e_{(3)}}, j = 1, 2, 3 \\
&\mathbf{e_{(i)}} \bullet \mathbf{e_{(j)}} = l_{ij}, i, j = 1, 2, 3
\end{aligned}$$

Since $\overline{\mathbf{e}}_{(j)}$ is a unit vector, it has a magnitude of unity.

Equation:

$$\mathbf{e_{(j)}} \bullet \mathbf{e_{(j)}} = 1 = l_{i(j)}l_{i(j)} = l_{1(j)}l_{1(j)} + l_{2(j)}l_{2(j)} + l_{3(j)}l_{3(j)}, j = 1, 2, 3$$

Also, the axis of a Cartesian system are orthorgonal.

Equation:

$$\begin{aligned}
&\mathbf{e_{(i)}} \bullet \mathbf{e_{(j)}} = \begin{cases} 0, \text{if} i \neq j \\ 1, \text{if} i = j \end{cases} \\
&\text{thus} \\
&\mathbf{e_{(i)}} \bullet \mathbf{e_{(j)}} = d_{ij}
\end{aligned}$$

Equation:

$$\begin{aligned}
\mathbf{e_{(i)}} \bullet \mathbf{e_{(j)}} &= l_{ki}l_{kj} = l_{1i}l_{1j} + l_{2i}l_{2j} + l_{3i}l_{3j}, i, j = 1, 2, 3 \\
&= \delta_{ij}
\end{aligned}$$

Vector Product

The *vector product* (or *cross product*) of two vectors, **a** and **b**, denoted as **a×b**, is a vector that is perpendicular to the plane of **a** and **b** such that **a**, **b**, and **a×b** form a right-handed system. (i.e., **a**, **b**, and **a×b** have the orientation of the thumb, first finger, and third finger of the right hand.) It has the following magnitude where θ is the angle between **a** and **b**.
Equation:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$$

The magnitude of the vector product is equal to the area of a parallelogram two of whose sides are the vectors **a** and **b**.

Since the vector product forms a right handed system, the product **b×a** has the same magnitude but opposite direction as **a×b**, i.e., the vector product is not commutative,
Equation:

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

The vector product of a vector with itself or with a parallel vector is zero or the null vector, i.e., **a**×**a**=**0**. A quantity that is the negative of itself is zero. Also, the angle between parallel vectors is zero and thus the sine is zero.

Consider the vector product of the unit vectors. They are all of unit length and mutually orthogonal so their vector products will be unit vectors. Remembering the right-handed rule, we therefore have
Equation:

$$\begin{aligned}
\mathbf{e}_{(2)} \times \mathbf{e}_{(3)} &= -\mathbf{e}_{(3)} \times \mathbf{e}_{(2)} = \mathbf{e}_{(1)} \\
\mathbf{e}_{(3)} \times \mathbf{e}_{(1)} &= -\mathbf{e}_{(1)} \times \mathbf{e}_{(3)} = \mathbf{e}_{(2)} \\
\mathbf{e}_{(1)} \times \mathbf{e}_{(2)} &= -\mathbf{e}_{(2)} \times \mathbf{e}_{(1)} = \mathbf{e}_{(3)}
\end{aligned}$$

The components of the vector product can be expressed in terms of the components of **a** and **b** and applying the above relations between the unit vectors.
Equation:

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \left(a_1\mathbf{e}_{(1)}+a_2\mathbf{e}_{(2)}+a_3\mathbf{e}_{(3)}\right)\times\left(b_1\mathbf{e}_{(1)}+b_2\mathbf{e}_{(2)}+b_3\mathbf{e}_{(3)}\right)\\ &= (a_2b_3-a_3b_2)\mathbf{e}_{(1)}+(a_3b_1-a_1b_3)\mathbf{e}_{(2)}+(a_1b_2-a_2b_1)\mathbf{e}_{(3)}\end{aligned}$$

The permutations of the indices and signs in the expression for the vector product may be difficult to remember. Notice that the expression is the same as that for the expansion of a determinate of the matrix,
Equation:

$$\begin{matrix}\mathbf{e}_{(1)} & \mathbf{e}_{(2)} & \mathbf{e}_{(3)}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{matrix}$$

Expansion of determinants are aided by the permutation symbol, ε_{ijk}.
Equation:

$$\varepsilon_{ijk}=\begin{cases}0, & \text{if any two of } i,j,k \text{ are the same}\\ +1, & \text{if } ijk \text{ is an even permutation of } 1,2,3\\ -1, & \text{if } ijk \text{ is an odd permutation of } 1,2,3\end{cases}$$

The expression for the vector product is now as follows.
Equation:

$$\mathbf{a}\times\mathbf{b}=\varepsilon_{ijk}a_ib_j\mathbf{e}_{(k)}$$

Velocity due to rigid body rotations

We will show that the velocity field of a rigid body can be described by two vectors, a translation velocity, $\mathbf{v}_{(t)}$, and an angular velocity, $\boldsymbol{\omega}$. A rigid body has the constraint that the distance between two points in the body does not change with time. The translation velocity is the velocity of a fixed point, *O*, in the body, e.g., the center of mass. Now consider a new reference frame (coordinate system) with the origin at point *O* that is translating with respect to the original reference frame with the velocity $\mathbf{v}_{(t)}$. The rotation of the body about *O* is defined by the angular velocity, $\boldsymbol{\omega}$, i.e., with a magnitude ω and a direction of the axis of rotation, **n**, such that the positive direction is the direction that a right handed screw advances when subject to the rotation, .$\boldsymbol{\omega}$=ω **n**. Consider a point *P* not on the axis of rotation, having coordinates **x** in the new reference frame. The velocity of *P* in the new reference frame has a magnitude equal to the product of ω and the radius of the point *P* from the axis of rotation. This radius is equal to the magnitude of **x** and the sine of the angle between **x** and $\boldsymbol{\omega}$, i.e., |**x**| sinθ. The velocity of point *P* in the new reference frame can be expressed as

Equation:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$$
$$|\mathbf{v}| = \omega\,|\mathbf{x}|\sin\theta$$

The velocity field of any point of the rigid body in the original reference frame is now

Equation:

$$\mathbf{v} = \mathbf{v}_{(t)} + \boldsymbol{\omega} \times \mathbf{x}$$

Since this equation is valid for any pair of points in the rigid body, the relative velocity Δ**v** between a pair of points separated by Δ**x** can be expressed as follows.

Equation:

$$\Delta\mathbf{v} = \boldsymbol{\omega} \times \Delta\mathbf{x}$$

Conversely, if the relative velocity between any pair of points is described by the above equation with the same value of angular velocity, then the motion is due to a rigid body rotation.

Triple scalar product

The triple scalar product is the scalar product of the first vector with the vector product of the other two vectors. It is denoted as (**abc**) or [**abc**].
Equation:

$$(\mathbf{abc}) = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$$

Recall that **b**×**c** has a magnitude equal to the area of a parallelogram with sides **b** and **c** and a direction normal to the plane of **b** and **c**. The scalar product of this normal vector and the vector **a** is equal to the altitude of the parallelepiped with a common origin and sides **a**, **b**, and **c**. The triple scalar product has a magnitude equal to the volume of a parallelepiped with a common origin and sides **a**, **b**, and **c**. The sign of the triple scalar product can be either positive or negative. If **a**, **b**, and **c** are coplanar, then the altitude of the parallelepiped is zero and thus the triple scalar product is zero.

The triple scalar product can be expressed in terms of the components by using the earlier definitions of the vector product and scalar product.
Equation:

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \varepsilon_{ijk} b_i c_j \mathbf{e}_{(k)} \\
\mathbf{a} &= a_m \mathbf{e}_{(m)} \\
\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) &= \varepsilon_{ijk} a_m b_i c_j \mathbf{e}_{(m)} \bullet \mathbf{e}_{(k)} = \varepsilon_{ijk} a_m b_i c_j \delta_{mk} = \varepsilon_{ijk} a_k b_i c_j \\
&= \varepsilon_{ijk} a_i b_j c_k
\end{aligned}$$

From the definition of the permutation symbol, the triple scalar product is unchanged by even permutations of **a**, **b**, and **c** but have the opposite algebraic sign for odd permutations. Also, if any two of **a**, **b**, and **c** are identical, then permutation of the two identical vectors results in a triple

scalar products that are identical and also opposite in sign. This implies that the triple scalar product is zero if two of the vectors are identical.

Triple vector product

The triple vector product of vectors **a**, **b**, and **c** results from the repeated application of the vector product, i.e., **a**×(**b**×**c**). Since **b**×**c** is normal to the plane of **b** and **c** and **a**×(**b**×**c**) is normal to **b**×**c, a**×(**b**×**c**) must be in the plane of **b** and **c**. It is left as an exercise to show that
Equation:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$$

Second order tensors

A second order tensor can be written as a 3×3 matrix.
Equation:

$$\mathbf{A} = \begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{matrix}$$

A tensor is a physical entity that is the same quantity in different coordinate systems. Thus a second order tensor is defined as an entity whose components transform on rotation of the Cartesian frame of reference as follows.
Equation:

$$A_{pq} = l_{ip} l_{jq} A_{ij}$$

If *Aij*=*Aji*the tensor is said to be *symmetric* and a symmetric tensor has only six distinct components. If *Aij*=-*Aij*the tensor is said to be *antisymmetric* and such a tensor is characterized by only three nonzero components for the diagonal terms, Ai_{i}, are zero. The tensor whose ij^{th} element is *Aji* is called

the transpose **A'** of **A**. The determinant of a tensor is the determinant of the matrix of its components.
Equation:

$$\det \mathbf{A} = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

Examples of second order tensors

A second order tensor we have already encountered is the *Kronecker delta* δ_{ij}. Of its nine components, the six off-diagonal components vanish and the three diagonal components are equal to unity. It transforms as a tensor upon transforming its components to a rotated frame of reference.
Equation:

$$\begin{aligned} \bar{\delta}_{pq} &= l_{ip} l_{jq} \delta_{ij} \\ &= l_{ip} l_{iq} \\ &= \delta_{pq} \end{aligned}$$

because of the orthogonality relation between the directional cosines l_{ij}. In fact, the components of δ_{ij}in all coordinates remain the same. δ_{ij} is called the isotropic tensor for that reason. The transport coefficients (e.g., thermal conductivity) of an <u>isotropic</u> medium can be expressed as a scalar quantity multiplying δ_{ij}.

If a and b are two vectors, the set of nine products, $a_i b_j = A_{ij}$, is a second order tensor, for
Equation:

$$\begin{aligned} A_{pq} &= a_p \bar{b}_q = l_{ip} a_i l_{jq} b_j = l_{ip} l_{jq} (a_i b_j) \\ &= l_{ip} l_{jq} A_{ij} \end{aligned}$$

An important example of this is the momentum flux tensor. If ρ is the density and **v** is the velocity, $\rho\, v_i$ is the ith component in the direction *Oi*. The rate at which this momentum crosses a unit area normal to *Oj* is the tensor, $\rho\, v_i\, v_j$.

Scalar multiplication and addition

If α is a scalar and **A** a second order tensor, the scalar product of α and **A** is a tensor α**A** each of whose components is α times the corresponding component of **A**.

The sum of two second order tensors is a second order tensor each of whose components is the sum of the corresponding components of the two tensors. Thus the ij^{th} component of **A**+**B** is $A_{ij} + B_{ij}$. Notice that the tensors must be of the same order to be added; a vector can not be added to a second order tensor. A linear combination of tensors results from using both scalar multiplication and addition. α**A** + ß**B** is the tensor whose ij^{th} component is αA_{ij} + ßB_{ij}. Subtraction may therefore be defined by putting $\alpha = 1$, $ß = -1$.

Any second order tensor can be represented as the sum of a symmetric part and an antisymmetric part. For
Equation:

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$$

and changing *i* and *j* in the first factor leaves it unchanged but changes the sign of the second. Thus,
Equation:

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A\prime}) + \frac{1}{2}(\mathbf{A} - \mathbf{A\prime})$$

represents **A** as the sum of a symmetric tensor and antisymmetric tensor.

Contraction and multiplication

As in vector operations, summation over repeated indices is understood with tensor operations. The operation of identifying two indices of a tensor and so summing on them is known as *contraction*. A_{ii} is the only contraction of A_{ij},
Equation:

$$A_{ii} = A_{11} + A_{22} + A_{33}$$

and this is no longer a tensor of the second order but a scalar, or a tensor of order zero. The scalar A_{ii} is known as the trace of the second order tensor **A**. The notation tr **A** is sometimes used. The contraction operation in computing the trace of a tensor **A** is analogous to the operation in the calculation the magnitude of a vector **a**, i.e.,
$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_1 a_1 + a_2 a_2 + a_3 a_3$

If **A** and **B** are two second order tensors, we can form 81 numbers from the products of the 9 components of each. The full set of these products is a fourth order tensor. Contracted products result in second order or zero order tensors. We will not have an occasion to use products of tensors in our course.

The product *Aij aj* of a tensor **A** and a vector **a** is a vector whose i^{th} component is $A_{ij}\, a_j$. Another possible product of these two is $A_{ij}\, a_I$. These may be written **A**·**a** and **a**·**A** , respectively. For example, the diffusive flux of a quantity is computed as the contracted product of the transport coefficient tensor and the potential gradient vector, e.g., $\mathbf{q} = -\mathbf{k} \cdot \boldsymbol{\nabla} T$

The vector of an antisymmetric tensor

We showed earlier that a second order tensor can be represented as the sum of a symmetric part and an antisymmetric part. Also, an antisymmetric tensor is characterized by three numbers. We will later show that the antisymmetric part of the velocity gradient tensor represents the local rotation of the fluid or body. Here, we will develop the relation between the

angular velocity vector, **ω**, introduced earlier and the corresponding antisymmetric tensor.

Recall that the relative velocity between a pair of points in a rigid body was described as follows.
Equation:

$$\Delta\mathbf{v} = \omega \times \Delta\mathbf{x}$$

We wish to define a tensor **Ω** that also can determine the relative velocity.
Equation:

$$\begin{aligned} \mathbf{\Delta v} &= \omega \times \mathbf{\Delta x} \\ &= \mathbf{\Delta x} \bullet \Omega \end{aligned}$$

The following relation between the components satisfies this relation.
Equation:

$$\begin{aligned} \Omega_{ij} &= \varepsilon_{ijk}\omega_k \\ \omega_k &= \tfrac{1}{2}\varepsilon_{ijk}\Omega_{ij} \end{aligned}$$

Written in matrix notation these are as follows.
Equation:

$$\omega = \begin{matrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{matrix}, \Omega = \begin{matrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{matrix}$$

The notation **vec Ω** is sometimes used for **ω**. In summary, an antisymmetric tensor is completely characterized by the vector, **vec Ω**.

Canonical form of a symmetric tensor

We showed earlier that any second order tensor can be represented as a sum of a symmetric part and an antisymmetric part. The symmetric part is determined by 6 numbers. We now seek the properties of the symmetric part. A theorem in linear algebra states that a symmetric matrix with real elements can be transformed by its eigenvectors to a diagonal matrix with real elements corresponding the eigenvalues. (see Appendix A of Aris.) If the eigenvalues are distinct, then the eigenvector directions are orthogonal. The eigenvectors determine a coordinate system such that the contracted product of the tensor with unit vectors along the coordinate axis is a parallel vector with a magnitude equal to the corresponding eigenvalue. The surface described by the contracted product of all unit vectors in this transformed coordinate system is an ellipsoid with axes corresponding to the coordinate directions.

The eigenvalues and the scalar invariants of a second order tensor can be determined from the characteristic equation.
Equation:

$$\det\left(A_{ij} - \lambda\delta_{ij}\right) = \psi - \lambda\Phi + \lambda^2\Theta - \lambda^3$$
$$\text{where}$$
$$\Theta = A_{11} + A_{22} + A_{33} = tr\mathbf{A}$$
$$\Phi = A_{22}A_{33} - A_{23}A_{32} + A_{33}A_{11} - A_{31}A_{13} + A_{11}A_{22} - A_{12}A_{21}$$
$$\psi = \det\mathbf{A}$$

Assignment 2.1

a. Relative velocity of points in a rigid body. If **x** and **y** are two points inside a rigid body that is translating and rotating, determine the relation between the relative velocity of these two points as a function of their relative positions. If **x** and **y** are points on a line parallel to the axis of rotation, what is their relative velocity? If **x** and **y** are points on opposite sides of the axis of rotation but with equal distance, r, what is their relative velocity? Draw diagrams.
b. Prove that: **a**•(**b**×**c**) = (**a**×**b**)•**c**
c. Show **a**•(**b**×**c**) vanishes identically if two of the three vectors are proportional of one another.

d. Show that if $\mathbf{a}$ is coplanar with $\mathbf{b}$ and $\mathbf{c}$, then $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$ is zero.

e. Prove that: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$

f. Prove that the contracted product of a tensor $\mathbf{A}$ and a vector $\mathbf{a}$, $\mathbf{A}\cdot\mathbf{a}$, transforms under a rotation of coordinates as a vector.

g. Show that you get the same result for relative velocity whether you use $\boldsymbol{\omega}$ or $\boldsymbol{\Omega}$ for the rotation of a rigid body.

Cartesian Vectors and Tensors: Their Calculus

Topics Covered in this Chapter

- Tensor functions of time-like variable
- Curves in space
- Line integrals
- Surface integrals
- Volume integrals
- Change of variables with multiple integrals
- Vector fields
- The vector operator -gradient of a scalar
- The divergence of a vector field
- The curl of a vector field
- Green's theorem and some of its variants
- Stokes' theorem
- The classification and representation of vector fields
- Irrotational vector fields
- Solenoidal vector fields
- Helmholtz' representation
- Vector and scalar potential

Reading assignment: Chapter 3 of Aris

Tensor functions of time-like variable

In the last chapter, vectors and tensors were defined as quantities with components that transform in a certain way with rotation of coordinates. Suppose now that these quantities are a function of time. The derivatives of these quantities with time will transform in the same way and thus are tensors of the same order. The most important derivatives are the velocity and acceleration.

Equation:

$$\mathbf{v}(t) = \dot{\mathbf{x}}(t), v_i = \frac{dx_i}{dt}$$
$$\mathbf{a}(t) = \ddot{\mathbf{x}}(t), a_i = \frac{d^2x_i}{dt^2}$$

The differentiation of products of tensors proceeds according to the usual rules of differentiation of products. In particular,
Equation:

$$\frac{d}{dt}(\mathbf{a} \bullet \mathbf{b}) = \frac{d\mathbf{a}}{dt} \bullet \mathbf{b} + \mathbf{a} \bullet \frac{d\mathbf{b}}{dt}$$
$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$

Curves in space

The trajectory of a particle moving is space defines a curve that can be defined with time as parameter along the curve. A curve in space is also defined by the intersection of two surfaces, but points along the curve are not associated with time. We will show that a natural parameter for both curves is the distance along the curve.

The variable position vector $\mathbf{x}(t)$ describes the motion of a particle. For a finite interval of t, say $a \leq t \leq b$, we can plot the position as a curve in space. If the curve does not cross itself (i.e., if $\mathbf{x}(t) \neq \mathbf{x}(t'), a \leq t < t' \leq b$) it is called *simple*; if $\mathbf{x}(a) = \mathbf{x}(b)$ the curve is closed. The variable t is now just a parameter along the curve that may be thought of as the time in motion of the particle. If t and t' are the parameters of two points, the cord joining them is the vector $\mathbf{x}(t') - \mathbf{x}(t)$. As $t \to t'$ this vector approaches $(t' - t)\dot{\mathbf{x}}(t)$ and so in the limit is proportional to $\dot{\mathbf{x}}(t)$. However the limit of the cord is the tangent so that $\dot{\mathbf{x}}(t)$ is in the direction of the tangent. If $v^2 = \dot{\mathbf{x}}(t) \bullet \dot{\mathbf{x}}(t)$ we can construct a *unit tangent vector* $\boldsymbol{\tau}$.
Equation:

$$\boldsymbol{\tau} = \dot{\mathbf{x}}(t)/v = \mathbf{v}/v$$

Now we will parameterize a curve with distance along the curve rather than time. If $\mathbf{x}(t)$ and $\mathbf{x}(t + dt)$ are two very close points,
Equation:

$$\mathbf{x}(t + dt) = \mathbf{x}(t) + dt\,\dot{\mathbf{x}}(t) + O\left(dt^2\right)$$

and the distance between them is
Equation:

$$\begin{aligned} ds^2 &= \{\mathbf{x}(t+dt)-\mathbf{x}(t)\} \bullet \{\mathbf{x}(t+dt)-\mathbf{x}(t)\} \\ &= \dot{\mathbf{x}}(t) \bullet \dot{\mathbf{x}}(t)\, dt^2 + O\left(dt^3\right) \end{aligned}$$

The arc length from any given point $t = a$ is therefore
Equation:

$$s(t) = \int_a^t \left[\dot{\mathbf{x}}\left(t'\right) \bullet \dot{\mathbf{x}}\left(t'\right)\right]^{1/2} dt'$$

s is the natural parameter to use on the curve, and we observe that
Equation:

$$\frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{\dot{\mathbf{x}}(t)}{v} = \tau$$

A curve for which a length can be so calculated is called *rectifiable*. From this point on we will regard s as the parameter, identifying t with s and letting the dot denote differentiation with respect to s. Thus
Equation:

$$\boldsymbol{\tau}(s) = \dot{\mathbf{x}}(s)$$

is the unit tangent vector. Let $\mathbf{x}(s), \mathbf{x}(s+ds)$, and $\mathbf{x}(s-ds)$ be three nearby points on the curve. A plane that passes through these three points is defined by the linear combinations of the cord vectors joining the points. This plane containing the points must also contain the vectors
Equation:

$$\frac{\mathbf{x}(s+ds)-\mathbf{x}(s)}{ds} = \dot{\mathbf{x}}(s) + O(ds)$$

and

$$\frac{\mathbf{x}(s+ds)-2\,\mathbf{x}(s)+\mathbf{x}(s-ds)}{ds^2} = \ddot{\mathbf{x}}(s) + O\left(ds^2\right)$$

Thus, in the limit when the points are coincident, the plane reaches a limiting position defined by the first two derivatives $\dot{\mathbf{x}}(s)$ and $\ddot{\mathbf{x}}(s)$. This limiting plane is called the *osculating plane* and the curve appears to lie in this plane in the intermediate neighborhood of the point. To prove this statement: (1) A plane is defined by the two vectors, $\dot{\mathbf{x}}(s)$ and $\ddot{\mathbf{x}}(s)$, if they are not co-linear. (2) The coordinates of the three points on the curve in the previous two equations are a linear combination of $\mathbf{x}(s), \dot{\mathbf{x}}(s)$ and $\ddot{\mathbf{x}}(s)$, thus they line in the plane.

Now $\dot{\mathbf{x}} = \boldsymbol{\tau}$ so $\ddot{\mathbf{x}} = \dot{\boldsymbol{\tau}}$ and since $\boldsymbol{\tau} \bullet \boldsymbol{\tau} = 1$,
Equation:

$$\frac{\mathbf{d}(\boldsymbol{\tau} \bullet \boldsymbol{\tau})}{ds} = 0 = \dot{\boldsymbol{\tau}} \bullet \boldsymbol{\tau} + \boldsymbol{\tau} \bullet \dot{\boldsymbol{\tau}} = 2\,\boldsymbol{\tau} \bullet \dot{\boldsymbol{\tau}}$$
$$\boldsymbol{\tau} \bullet \dot{\boldsymbol{\tau}} = 0$$

so that the vector $\dot{\boldsymbol{\tau}}$ is at right angles to the tangent. Let $1/\rho$ denote the magnitude of $\dot{\boldsymbol{\tau}}$.
Equation:

$$\dot{\boldsymbol{\tau}} \bullet \dot{\boldsymbol{\tau}} = \frac{1}{\rho^2}$$
$$\text{and}$$
$$\nu = \rho\,\dot{\boldsymbol{\tau}}$$

Then $\boldsymbol{\nu}$ is a unit normal and defines the direction of the so-called principle normal to the curve.

To interpret ρ , we observe that the small angle $d\theta$ between the tangents at s and $s + ds$ is given by
Equation:

$$\cos d\theta = \tau(s) \bullet \tau(s + ds)$$
$$1 - \frac{1}{2} d\theta^2 + \ldots = \tau \bullet \tau + \tau \bullet \dot{\tau}\, ds + \frac{1}{2} \tau \bullet \ddot{\tau}\, ds^2 + \ldots$$
$$= 1 - \frac{1}{2} \dot{\tau} \bullet \dot{\tau}\, ds^2 + \ldots$$

since $\tau \bullet \dot{\tau} = 0$ and so $\tau \bullet \ddot{\tau} + \dot{\tau} \bullet \dot{\tau} = 0$. Thus,
Equation:

$$\rho = \frac{ds}{d\theta}$$

is the reciprocal of the rate of change of the angle of the tangent with arc length, i.e., ρ is the radius of curvature. Its reciprocal $1/\rho$ is the curvature, $\kappa \equiv |d\theta/ds| = 1/\rho$.

A second normal to the curve may be taken to form a right-hand system with τ and ν . This is called the unit binormal,
Equation:

$$\boldsymbol{\beta} = \tau \times \nu$$

Line integrals

If $F(x)$ is a function of position and C is a curve composed of connected arcs of simple curves, $\mathbf{x} = \mathbf{x}(t), a \leq t \leq b$ or $\mathbf{x} = \mathbf{x}(s), a \leq s \leq b$, we can define the integral of F along C as
Equation:

$$\int_C F(\mathbf{x})dt = \int_a^b F[\mathbf{x}(t)]dt$$
$$\text{or}$$
$$\int_C F(\mathbf{x})ds = \int_a^b F[\mathbf{x}(t)]\{\dot{\mathbf{x}}\,(t) \bullet \dot{\mathbf{x}}\,(t)\}^{1/2}dt$$

Henceforth, we will assume that the curve has been parameterized with respect to distance along the curve, s.

The integral is from a to b. If the integral is in the opposite direction with opposite limits, then the integral will have the same magnitude but opposite

sign. If $\mathbf{x}(a) = \mathbf{x}(b)$, the curve C is closed and the integral is sometimes written
Equation:

$$\oint_C F[\mathbf{x}(s)]ds$$

If the integral around any simple closed curve vanishes, then the value of the integral from any pair of points a and b is independent of path. To see this we take any two paths between a and b, say C_1 and C_2, and denote by C the closed path formed by following C_1 from a to b and C_2 back from b to a.
Equation:

$$\begin{aligned} \oint_C F\,ds &= \left[\left[\int_a^b F\,ds\right]\right]_{C_1} + \left[\left[\int_b^a F\,ds\right]\right]_{C_2} \\ &= \left[\left[\int_a^b F\,ds\right]\right]_{C_1} - \left[\left[\int_a^b F\,ds\right]\right]_{C_2} \\ &= 0 \end{aligned}$$

If $\mathbf{a}(\mathbf{x})$ is any vector function of position, $\mathbf{a} \bullet \tau$ is the projection of $\mathbf{a}$ tangent to the curve. The integral of $\mathbf{a} \bullet \tau$ around a simple closed curve C is called the *circulation* of a around C.
Equation:

$$\oint_C \mathbf{a} \bullet \boldsymbol{\tau}\,ds = \oint_C a_i[x_1(s), x_2(s), x_3(s)]\,\tau_i\,ds$$

We will show later that a vector whose circulation around any simple closed curve vanishes is the gradient of a scalar.

Surface integrals

Many types of surfaces and considered in transport phenomena. Most often the surfaces are the boundaries of volumetric region of space where boundary conditions are specified. The surfaces could also be internal boundaries where

the material properties change between two media. Finally the surface itself may the subject of interest, e.g. the statics and dynamics of soap films.

A proper mathematical treatment of surfaces requires some definitions. A *closed surface* is one which lies within a bounded region of space and has an inside and outside. If the normal to the surface varies continuously over a part of the surface, that part is called *smooth*. The surface may be made up of a number of subregions, which are smooth and are called *piece-wise smooth*. A closed curve on a surface, which can be continuously shrunk to a point, is called *reducible*. If all closed curves on a surface are reducible, the surface is called *simply connected*. The sphere is simply connected but a torus is not.

If a surface is not closed, it normally has a space curve as its boundary, as for example a hemisphere with the equator as boundary. It has two sides if it is impossible to go from a point on one side to the other along a continuous curve that does not cross the boundary curve. The surface is sometimes called the *cap* of the space curve.

If S is a piece-wise smooth surface with two sides in three-dimensional space, we can divide it up into a large number of small surface regions such that the dimensions of the regions go to zero as the number of regions go to infinity. If the regions fill the surface and are not overlapping, then sum of the areas of the regions is equal to the area of the surface. If the function, F is defined on the surface, it can be evaluated for some point of each subregion of the surface and the sum $\Sigma F \Delta S$ computed. The limit as the number of regions go to infinity and the dimensions of the regions go to zero is *surface integral* of F over S.
Equation:

$$\lim \sum F \, \Delta S = \iint_S F \, dS$$

The traditional symbol of the double integral is retained because if the surface is a plane or the surface is projected on to a plane, then Cartesian coordinates can be defined such that the surface integral is a double integral of the two coordinates in the plane. Also, two surface coordinates can define a surface and the double integration is over the surface integrals.

In transport phenomena the surface integral usually represents the flow or flux of a quantity across the surface and the function F is the normal component of a vector or the contracted product of a tensor with the unit normal vector. Thus one needs to know the direction of the normal in addition to the differential area to calculate the surface integral. Consider the case of a surface defined as a function of two surface coordinates.

Equation:

$$\mathbf{x}\left(u_1, u_2\right) = \mathbf{f}\left(u_1, u_2\right) \mathrm{on} S$$
$$\mathbf{n}\, dS = \left(\frac{\partial \mathbf{f}}{\partial u_1} \times \frac{\partial \mathbf{f}}{\partial u_2}\right) du_1\, du_2$$

To see how we arrive at this result, recall the partial derivatives of the coordinates of a curve with respect to a parameter is a vector that is tangent to the curve. The magnitude is

Equation:

$$\begin{aligned} \frac{\partial \mathbf{f}}{\partial u_i} &= \left(\frac{\partial \mathbf{f}}{\partial u_i} \bullet \frac{\partial \mathbf{f}}{\partial u_i}\right)^{1/2} \\ &= \left[\left(\frac{\partial f_1}{\partial u_i}\right)^2 + \left(\frac{\partial f_2}{\partial u_i}\right)^2 + \left(\frac{\partial f_3}{\partial u_i}\right)^2\right]^{1/2} \\ &= \left(\frac{ds}{du_i}\right)_{u_j} \end{aligned}$$

The vector product has a magnitude equal to the product of the magnitudes and the sine of the angle between the vectors. This gives us the area of a parallelogram corresponding to the area of the differential region.

Equation:

$$dS = \left(\frac{ds}{du_1}\right)_{u2} \left(\frac{ds}{du_2}\right)_{u1} \sin\theta\, du_1\, du_2$$

The two tangent vectors in the direction of the surface coordinates lie in the tangent plane of the surface. Thus the direction of the vector product is

perpendicular to the surface. Inward or outward direction for the normal has not yet been specified and will be determined by the sign.

Volume integrals

The volume integral of a function F over a volumetric region of space V is the limit of the sum of the products of the volume of small volumetric subregions of V and the function F evaluated somewhere within each subregion.
Equation:

$$\iiint\limits_V F(\mathbf{x})\, dV = \lim \sum F\, \Delta V$$

Change of variables with multiple integrals

In Cartesian coordinates the elements of volume dV is simply the volume of a rectangular parallelepiped of sides dx_1, dx_2, dx_3 and so
Equation:

$$dV = dx_1\, dx_2\, dx_3$$

Suppose, however, that it is convenient to describe the position by some other coordinates, say ξ_1, ξ_2, ξ_3. We may ask what volume is to be associated with the three small changes $d\xi_1$, $d\xi_2$, $d\xi_3$.

The change of coordinates must be given by specifying the Cartesian point x that is to correspond to a given set ξ_1, ξ_2, ξ_3, by
Equation:

$$x_i = x_i\left(\xi_1, \xi_2, \xi_3\right)$$

Then by partial differentiation the small differences corresponding to a change $d\xi_i$ are

Equation:

$$dx_i = \frac{\partial x_i}{\partial \xi_j}\, d\xi_j$$

Let $d\mathbf{x}^{(j)}$ be the vectors with the components $(\delta x_i/\delta \xi_j)d\xi_j$ for $j = 1, 2, and 3$. Then the volume element is
Equation:

$$\begin{aligned} dV &= d\mathbf{x}^{(1)} \bullet \left(d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)}\right) \\ &= \varepsilon_{ijk}\, \tfrac{\partial x_i}{\partial \xi_1} d\xi_1\, \tfrac{\partial x_j}{\partial \xi_2} d\xi_2\, \tfrac{\partial x_k}{\partial \xi_3} d\xi_3 \\ &= J\, d\xi_1\, d\xi_2\, d\xi_3 \end{aligned}$$

where
Equation:

$$J = \frac{\partial(x_1,\, x_2,\, x_3)}{\partial(\xi_1,\, \xi_2,\, \xi_3)} = \varepsilon_{ijk}\, \frac{\partial x_i}{\partial \xi_1}\, \frac{\partial x_j}{\partial \xi_2}\, \frac{\partial x_k}{\partial \xi_3}$$

is called the *Jacobian* of the transformation of variables

Vector fields

When the components of a vector or tensor depend on the coordinates we speak of a vector or tensor field. The flow of a fluid is a perfect realization of a vector field for at each point in the region of flow we have a velocity vector $\mathbf{v}(\mathbf{x})$. If the flow is unsteady then the velocity depends on the time as well as position, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$.

Associated with any vector field $\mathbf{a}(\mathbf{x})$ are its *trajectories*, which is the name given to the family of curves everywhere tangent to the local vector $\mathbf{a}$. They are solutions of the simultaneous equations
Equation:

$$\frac{d\mathbf{x}}{ds} = \mathbf{a}\left(\mathbf{x}\right); \mathrm{thatis} \frac{dx_i}{ds} = a_i\left(x_1,\, x_2,\, x_3\right)$$

Where s is a parameter along the trajectory. (It will be arc length if $\mathbf{a}$ is always a unit vector.) Streamlines of a steady flow are a realization of these trajectories. For a time dependent vector field the trajectories will also be time dependent. If C is any closed curve in the vector field and we take the trajectories through all points of C, they describe a surface known as the *vector tube* of the field. For flow fields, it is called a *stream tube*.

The vector operator ∇-gradient of a scalar

The symbol ∇ (enunciated as "del") is used for the symbolic vector operator whose i^{th} component is $\delta/\delta x_i$. Thus if ∇ operates on a scalar function of position $\phi(\mathbf{x})$ it produces a vector $\nabla\phi$ with components $\delta\phi/\delta x_i$.
Equation:

$$\frac{\partial\phi}{\partial x_j} = \frac{\partial\phi}{\partial x_i}\frac{\partial x_i}{\partial x_j} = l_{ij}\,\frac{\partial\phi}{\partial x_i}$$

Equation:

$$\mathrm{grad}\,\phi = \nabla\phi = \phi_{,i} = \mathbf{e}_{(1)}\,\frac{\partial\phi}{\partial x_1} + \mathbf{e}_{(2)}\,\frac{\partial\phi}{\partial x_2} + \mathbf{e}_{(3)}\,\frac{\partial\phi}{\partial x_3}$$

since $a_j = l_{ij}\,a_i$ so $\nabla\phi$ is a vector. The suffix notation, i for the partial derivative with respect to x_i is a very convenient one and will be taken over for the generalization of this operation that must be made for non-Cartesian frame of reference. The notation "grad" for ∇ is often used and referred to as the gradient operator. Thus grad ϕ is the vector which is the gradient of the scalar. The gradient operator can also operate on higher order tensors and the operation raises the order by one. Thus the gradient of a vector $\mathbf{a}$ is grad $\mathbf{a}$, $\nabla\mathbf{a}$, or in component notation $a_{i,j}$. ∇ is sometimes written $\delta/\delta\mathbf{x}$ and can be expanded as the vector operator

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Equation:

$$\nabla \equiv \mathbf{e}_{(i)}\,\frac{\partial}{\partial x_i} = \mathbf{e}_{(1)}\,\frac{\partial}{\partial x_1} + \mathbf{e}_{(2)}\,\frac{\partial}{\partial x_2} + \mathbf{e}_{(3)}\,\frac{\partial}{\partial x_3}$$

since $a_j = l_{ij}\,a_i$ so $\nabla\phi$ is a vector.

Equation:

$$\begin{aligned} d\phi &= \tfrac{\partial\phi}{\partial x_i}\,dx_i \\ &= \tfrac{\partial\phi}{\partial x_1}\,dx_1 + \tfrac{\partial\phi}{\partial x_2}\,dx_2 \tfrac{\partial\phi}{\partial x_3}\,dx_3 \\ &= \left(\mathbf{e}_{(1)}\,\tfrac{\partial\phi}{\partial x_1} + \mathbf{e}_{(2)}\,\tfrac{\partial\phi}{\partial x_2} + \mathbf{e}_{(3)}\,\tfrac{\partial\phi}{\partial x_3}\right) \bullet \left(\mathbf{e}_{(1)}\,dx_1 + \mathbf{e}_{(2)}\,dx_2 + \mathbf{e}_{(3)}\,dx_3\right) \\ &= \nabla\phi \bullet d\mathbf{x} \end{aligned}$$

The unit vector in the direction of $d\mathbf{x}$ is $\mathbf{u} = d\mathbf{x}/ds$. The derivative of ϕ in the direction of $\mathbf{u}$ is

Equation:

$$\frac{d\phi}{ds} = \nabla\phi \bullet \mathbf{u} = |\nabla\phi|\ \cos\theta$$

If $\phi(x) = c$ is a surface, then $\nabla\phi$ is normal to the surface. To prove this, let $d\mathbf{x}$ be a differential distance on the surface. The differential of ϕ along $d\mathbf{x}$ is zero for any $d\mathbf{x}$ on the surface. This implies that the scalar product of $\nabla\phi$ with any vector on the surface is zero or that $\nabla\phi$ has zero component or projection on the tangent plane and thus $\nabla\phi$ is normal to the surface. Also, since $\nabla\phi$ is

normal to the surface, the derivative of ϕ is a maximum in the direction normal to the surface.

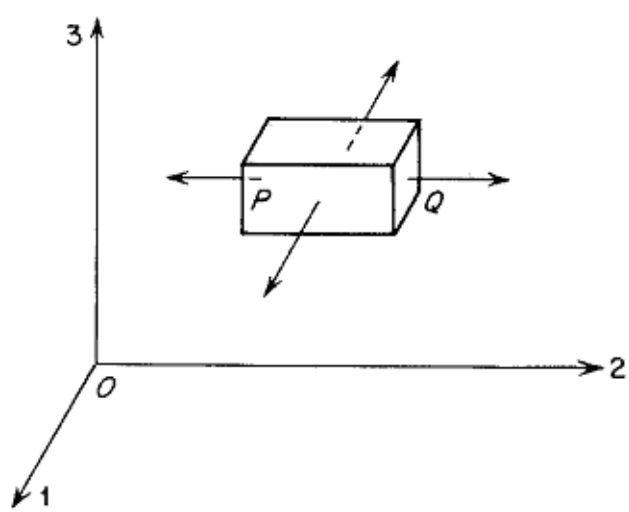
The divergence of a vector field

The symbolic scalar or dot product of the operator ∇ and a vector is called the *divergence* of the vector field. Thus for any differentiable vector field $\mathbf{a}(\mathbf{x})$ we write

Equation:

$$div\,\mathbf{a} = \nabla \bullet \mathbf{a} = a_{i,i} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$$

The divergence is a scalar because it is the scalar product and because it is the contraction of the second order tensor $a_{i,j}$.



We will now demonstrate why the $\nabla\bullet$ operation on a vector field is called the divergence. Suppose that an elementary parallelepiped is set up with one corner P at x_1, x_2, x_3 and the diagonally opposite one Q at $x_1 + dx_1$, $x_2 + dx_2$, $x_3 + dx_3$ as shown in Fig. 3.7. The outward unit normal to the face through Q which is perpendicular to *01* is $\mathbf{e}_{(1)}$, whereas the outward normal to the parallel

face through P is $-\mathbf{e}_{(1)}$. Suppose $\mathbf{a}(\mathbf{x})$ is a differentiable flux vector field. We are going to compute the net flux of $\mathbf{a}$ across the bounding surfaces of the parallelepiped. The value of the normal component of $\mathbf{a}$ at some point on the two faces perpendicular to the 01 direction are

Equation:

$$\begin{aligned}
&a_1\left(x_1, x_2, x_3\right)\mathrm{and} a_1\left(x_1+d x_1, x_2, x_3\right) \\
&\text{where} \\
&\mathrm{x}_2 \leqslant x_2 \leqslant x_2+d x_2 \mathrm{and} \mathrm{x}_3 \leqslant x_3 \leqslant x_3+d x_3
\end{aligned}$$

Thus if $\mathbf{n}$ denotes the outward normal and dS is the area $dx_2 dx_3$ of these two faces, we have a contribution from them to the surface integral $\oint_S \mathbf{an}\, dS$ of

Equation:

$$\begin{aligned}
\iint_{01\mathrm{faces}} \mathbf{a} \bullet \mathbf{n}\, dS &= \left[a_1\left(x_1+d x_1, x_2, x_3\right)-a_1\left(x_1, x_2, x_3\right)\right] d x_2\, d x_3 \\
&= \frac{\partial a_1}{\partial x_1} d x_1\, d x_2\, d x_3+O\left(d x^4\right)
\end{aligned}$$

where $O\left(dx^4\right)$ denotes terms proportional to fourth power of dx. Similar terms with $\delta a_2 / \delta x_2$ and $\delta a_3 / \delta x_3$ will be given by contributions of the other faces so that for the whole parallelepiped whose volume $dV=dx_1 dx_2 dx_3$ we have

Equation:

$$\frac{1}{d V} \iint_S \mathbf{a} \bullet \mathbf{n}\, d S=\frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}+O(d x)$$

If we let the volume shrink to zero we have

Equation:

$$\lim_{d V \rightarrow 0} \frac{1}{d V} \iint_S \mathbf{a} \bullet \mathbf{n}\, d S=\nabla \bullet \mathbf{a}$$

If $\mathbf{a}$ is a flux, then the surface integral is the net flux of $\mathbf{a}$ out of the volume. In particular, let $\mathbf{a}$ be the fluid velocity, which can be thought as a volumetric flux. Then the divergence of velocity is the volumetric expansion per unit volume. A vector field with identically zero divergence is called *solenoidal*. An incompressible fluid has a solenoidal velocity field. If the flux field of a certain property is solenoidal there is no generation of that property within the field, for all that flows into an infinitesimal element flows out again.

If $\mathbf{a}$ is the gradient of a scalar function $\nabla\phi$, its divergence is called the *Laplacian* of ϕ.
Equation:

$$\nabla^2\phi = \nabla \bullet (\nabla\phi) = \phi_{,ii} = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}$$

A function that satisfies Laplace's equation $\nabla^2\phi = 0$ is called a *potential function* or a *harmonic function*. An irrotational, incompressible flow field has a velocity that is the gradient of a flow potential. Also, the steady-state temperature field in a homogeneous solid and the steady state pressure distribution of a single fluid phase flowing in porous media are solutions of Laplace's equation. In two dimensions the solutions of Laplace's equation can be found through the use of complex variables.

If $\mathbf{A}$ is a tensor, the notation $div\mathbf{A}$ or $\nabla \bullet \mathbf{A}$ is sometimes used for the vector $A_{ij,i}$. The index notation is preferred for tensors.

The curl of a vector field

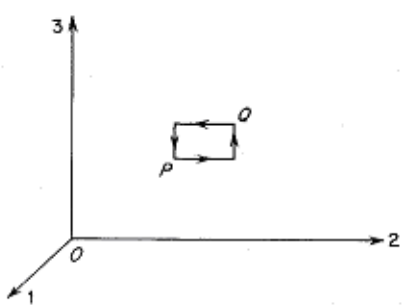
The symbolic vector product or cross product of the vector operator ∇ and a vector field $\mathbf{a}(\mathbf{x})$ is called the *curl* of the vector field. It is the vector
Equation:

$$\nabla \times \mathbf{a} = curl\,\mathbf{a} = \varepsilon_{ijk}\, a_{k,j}\, \mathbf{e}_{(i)}$$

That this definition is a combination of the previously definitions for the ∇ operator and the cross product can be seen be carrying out the operations.

Equation:

$$\begin{aligned}
\nabla \times \mathbf{a} &= \left(\mathbf{e}_{(1)}\,\frac{\partial}{\partial x_1} + \mathbf{e}_{(2)}\,\frac{\partial}{\partial x_2} + \mathbf{e}_{(3)}\,\frac{\partial}{\partial x_3}\right) \times \left(\mathbf{e}_{(1)}\,a_1 + \mathbf{e}_{(2)}\,a_2 + \mathbf{e}_{(3)}\,a_3\right) \\
&= \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathbf{e}_{(1)} + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\mathbf{e}_{(2)} + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathbf{e}_{(3)} \\
&\quad \text{also} \\
\nabla \times \mathbf{a} &= \varepsilon_{ijk}\,\frac{\partial a_j}{\partial x_i}\,\mathbf{e}_{(k)} = \varepsilon_{ijk}\,a_{j,i}\,\mathbf{e}_{(k)} = \varepsilon_{kji}\,a_{j,k}\,\mathbf{e}_{(i)} = \varepsilon_{jki}\,a_{k,j}\,\mathbf{e}_{(i)} \\
&= \varepsilon_{ijk}\,a_{k,j}\,\mathbf{e}_{(i)} Q.E.D.
\end{aligned}$$



The connection between the curl of a vector field and the rotation of the vector field (it is called rot $\mathbf{a}$ in some older texts) can be illustrated by calculating the circulation of the vector field around a closed curve. Consider a elementary rectangle in the 023 plane normal to 01 with one corner P at (x_1, x_2, x_3) and the diagonally opposite one Q at (x_1, $x_2 + dx_2$, $x_3 + dx_3$) as shown in Fig. 3.8. We wish to calculate the line integral or circulation around this elementary closed curve of $\mathbf{a} \bullet \mathbf{t} ds$, where $\mathbf{t}$ is the unit tangent to the curve. Now the line through P parallel to 03 has tangent $-\mathbf{e}_{(3)}$ and the parallel side through Q has tangent $\mathbf{e}_{(3)}$, and each is of length dx_3. Accordingly, they contribute to $\mathbf{a} \bullet \mathbf{t} ds$ an amount

Equation:

$$\left[a_3\left(x_1, x_2 + dx_2, x_3\right) - a_3\left(x_1, x_2, x_3\right)\right] dx_3 = \frac{\partial a_3}{\partial x_2}\,dx_2\,dx_3 + O\left(dx^3\right)$$

Similarly, from the other two sides, there is a contribution,

Equation:

$$-\frac{\partial a_2}{\partial x_3}\,dx_3\,dx_2 + O\left(dx^3\right)$$

Thus writing $dA = dx_2\ dx_3$, we have
Equation:

$$\frac{1}{dA}\oint\limits_{023}\mathbf{a}\bullet\mathbf{t}ds = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) + O\left(dx\right)$$

and in the limit
Equation:

$$\lim\limits_{dA\to 0}\frac{1}{dA}\oint\limits_{023}\mathbf{a}\bullet\mathbf{t}ds = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) = \left(\nabla\times\mathbf{a}\right)_1 = \left(\nabla\times\mathbf{a}\right)\bullet\mathbf{n}$$

The suffix 023 has been put on the integral sign to show that the line integral in a 023 plane, and the last equation shows that the circulation in the $O23$ plane is equal to the component of the curl in the $O1$ direction. This correspondence between the curl and circulation gives physical meaning the curl of a vector field. It is a measure of the circulation or rotation of the motion. There is a direction associated with circulation, rotation, and curl. If the circulation around a closed curve is in the direction of the closed fingers of the right hand, then the curl is in the direction of the thumb.

A vector field $\mathbf{a}$ for which $\nabla\times\mathbf{a} = 0$ is called *irrotational* because the circulation about any closed curve vanishes.
Equation:

$$\nabla \bullet (\nabla \times \mathbf{a}) = 0$$
$$\nabla \times (\nabla \phi) = 0$$
$$\nabla \times (\phi \mathbf{a}) = (\nabla \phi) \times \mathbf{a} + \phi \nabla \times \mathbf{a}$$
$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \bullet \mathbf{b}) - \mathbf{b}(\nabla \bullet \mathbf{a}) + (\mathbf{b} \bullet \nabla)\mathbf{a} - (\mathbf{a} \bullet \nabla)\mathbf{b}$$
$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \bullet \mathbf{a}) - \nabla^2 \mathbf{a}$$

The first of these states that if a vector field $\mathbf{b}$ is the curl of a vector, i.e., $\mathbf{b} = \nabla \times \mathbf{a}$ then the vector field $\mathbf{b}$ is solenoidal. The second states that if a vector field $\mathbf{b}$ is equal to the gradient of a scalar, i.e., $\mathbf{b} = \nabla \phi$ then the vector field $\mathbf{b}$ is irrotational. The last identity has the Laplacian operator $\nabla^2 = \nabla \bullet \nabla$ operating on a vector. The result is a vector whose components are equal to the Laplacian of the components, if the coordinates are Cartesian. This may not be the case in curvilinear coordinates.

Green's theorem and some of its variants

The divergence theorem, also called the Gauss' theorem, or Green's theorem equates the volume integral of the divergence of a vector field $\mathbf{a}(\mathbf{x})$ to the surface integral of the normal component of the vector.
Equation:

$$\iiint_V \nabla \bullet \mathbf{a}\, dV = \oiint_S \mathbf{a} \bullet \mathbf{n}\, dS$$

During the discussion of the divergence of a vector field, we showed that the above relation holds for an infinitesimal volume. Suppose now that a macroscopic volume of space is a composite of the infinitesimal regions. The total volume integral is the sum of the infinitesimal volume integrals. However, the contribution of $\mathbf{a} \bullet \mathbf{n} dS$ from the touching faces of two adjacent elements of volume are equal in magnitude but opposite in sign since the outward normal points in opposite directions. Thus in a summation of $\mathbf{a} \bullet \mathbf{n} dS$, the only terms that survive are those on the outer surface S, i.e., the surface integral is over the exterior surfaces of the macroscopic region. Q.E.D.

If $\mathbf{a} = \nabla\phi$ we have
Equation:

$$\iiint_V \nabla^2\phi \, dV = \iiint_V \nabla \bullet \nabla\phi \, dV = \oiint_S \nabla\phi \bullet \mathbf{n} \, dS = \oiint_S \frac{\partial\phi}{\partial n} \, dS$$

where $\delta\phi/\delta n$ denotes the derivative in the direction of the outward normal. If the scalar ϕ is temperature, this equation says that at steady-state, the integral of the net sources of heat in the volume is equal to the flux across the external surfaces.

Stokes' theorem

On a surface let C be the curve or a finite number of curves forming the complete boundary of an area S. We assume that the surface is two-sided and that S can be resolved into a finite number of regular elements. Choose a positive side of S and let the positive direction along C be that in which an observer on the positive side must move along the boundary if he is to have the area S always on his left. At each regular point on the surface let $\mathbf{n}$ be the unit normal drawn toward the positive side. Let $\mathbf{a}$ and its first derivatives be continuous on $\mathbf{S}$. Stokes' theorem states that the circulation around a closed curve is equal to the surface integral of the normal component of the curl.
Equation:

$$\oint_C \mathbf{a} \bullet \mathbf{t} \, ds = \iint_S (\nabla \times \mathbf{a}) \bullet \mathbf{n} \, dS$$

We showed earlier the circulation around an infinitesimal, closed curve was equal to the normal component of the curl multiplied by the area of the enclosed surface. We will extend the earlier result for an infinitesimal closed curve enclosing an infinitesimal surface to a macroscopic curve and surface. The macroscopic surface will be subdivided into a composite of many infinitesimal regions where the earlier result apply. The summation of the normal component of the curl multiplied by the area of the element is equal to

the surface integral of the normal component of the curl. However, the quantity $\mathbf{a} \bullet \mathbf{t} ds$ from the touching sides of two adjacent surface elements have equal magnitude but opposite sign since the direction of the line integrals are in opposite directions. Thus in the summation of the circulations, the only terms that survive are the contribution of the external bounding curve, i.e., the circulation is around the exterior curve C bounding the surface S. Q.E.D.

The classification and representation of vector fields

We mentioned earlier that a solenoidal vector field is one where $\nabla \bullet \mathbf{a} = 0$ everywhere and an irrotational vector field is one where $\nabla \times \mathbf{a} = 0$ everywhere. A vector field that is the gradient of a scalar $\mathbf{a} = \nabla \phi$ is irrotational. If a vector field is both irrotational and solenoidal it is the gradient of a harmonic function, where $\nabla \bullet (\nabla \phi) = \nabla^2 \phi = 0$. It can be proven that if a vector field is both irrotational and solenoidal, it is uniquely determined in a volume V if it is specified over S, the surface of V. There other types of named vector fields are discussed by Aris.

Irrotational vector fields

The vector field $\mathbf{a}$ is irrotational if its curl vanishes everywhere. By Stokes' theorem the circulation around any closed curve also vanishes. Also, an irrotational vector field can be expressed as the gradient of a scalar.
Equation:

$$\begin{aligned} \nabla \times \mathbf{a} &= 0 \\ \oint_C \mathbf{a} \bullet \mathbf{t} ds &= 0 \quad \mathbf{a} \text{ an irrotational field} \\ \mathbf{a} &= \nabla \phi \end{aligned}$$

The velocity field of motions where the viscous effects are insignificant compared to inertial effects and the flow is initially irrotational can be approximated as an irrotational velocity field.

Solenoidal vector fields

A solenoidal vector field is defined as one in which the divergence vanishes. This implies that the flux across a closed surface must also vanish. A vector identity states that the divergence of the curl of a vector is zero. Thus a continuously differentiable solenoidal vector field has the following three equivalent characteristics.

Equation:

$$\begin{aligned} \nabla \bullet \mathbf{a} &= 0 \\ \oint_S \mathbf{a} \bullet \mathbf{n} dS &= 0 \quad \mathbf{a} \text{ a solenoidal field} \\ \mathbf{a} &= \nabla \times \mathbf{A} \end{aligned}$$

The velocity field of motions where the effects of compressibility are insignificant can be approximated as a solenoidal vector field. The surface integral of velocity vanishing over any closed surface means that the net volumetric flow across closed surfaces is zero. Incompressible flow fields can be expressed as the curl of a vector potential. Two-dimensional, incompressible flows have only one nonzero component of the vector potential and this is identified as the stream function.

Helmholtz' representation

We found that an irrotational vector is the gradient of a scalar potential and a solenoidal vector is the curl of a vector potential. Here we show that any vector field with sufficient continuity is divisible into irrotational and solenoidal parts, and so is expressible in terms of a scalar and a vector potential. The fundamental problem in the analysis of a vector field is the determination of these potentials and their expression in terms of the essential characteristics of the vector, namely divergence, curl, discontinuities, and boundary values. For when the potentials are known the vector itself can be determined by differentiation. The following analysis is taken from H. B. Phillips, *Vector Analysis*, John Wiley & Sons, 1933. The following nomenclature will differ somewhat in that the vector is expressed as the negative of the gradient of a scalar. Also, the vector field of interest will be denoted as $mathbfF$. Bold face

capital letters will also be used for other vector quantities. Also the equations have the 4π factor of electromagnetism in mks units rather than the factors ε_o and $\varepsilon_o\ c^2$ of the SI units.

Let V be a region of space where the vector field F has piecewise continuous second derivatives, S_1 be surfaces of discontinuity of F, and S be the bounding surface of V. The Helmholtz's theorem states that F can be expressed in terms of the potentials.
Equation:

$$\begin{aligned}
&\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A} \\
&\text{where} \\
&\phi\left(\mathbf{x}_P\right) = \iiint\limits_{\mathrm{V}} \frac{\rho dV}{r} + \iint\limits_{S_1} \frac{\sigma dS}{r} - \frac{1}{4\pi} \oint\limits_{S} \frac{\mathbf{n} \bullet \mathbf{F} dS}{r} \\
&\mathbf{A}\left(\mathbf{x}_P\right) = \iiint\limits_{V} \frac{\mathbf{I} dV}{r} + \iint\limits_{S_1} \frac{\mathbf{J} dS}{r} - \frac{1}{4\pi} \oint\limits_{S} \frac{\mathbf{n} \times \mathbf{F} dS}{r} \\
&\nabla \bullet \mathbf{F} = -\nabla^2\phi = 4\pi\rho \\
&\nabla \times \mathbf{F} = \nabla \times \left(\nabla \times \mathbf{A}\right) = 4\pi\mathbf{I} \\
&\mathbf{n} \bullet \Delta\mathbf{F} = 4\pi\sigma \\
&\mathbf{n} \times \Delta\mathbf{F} = 4\pi\mathbf{J} \\
&r = \left|\mathbf{x}_P - \mathbf{x}_Q\right| \mathrm{where} \mathbf{x}_Q \mathrm{iscoordinateofintegrand}
\end{aligned}$$

The vectors $\mathbf{I}$ and $\mathbf{J}$ are not arbitrary. They are subject to the equation of continuity $\nabla \bullet \mathbf{A} = 0$. The effect of this condition is to make $\mathbf{I}$ and $\mathbf{J}$ behave like space and surface currents of something which is nowhere created or destroyed. In the electromagnetic field they usually represent currents of electricity. In hydrodynamic fields they represent vorticity. If $\mathbf{A}$ is everywhere solenoidal, the following three equations must then be everywhere satisfied.
Equation:

$$\begin{aligned}
&\nabla \bullet \mathbf{I} = 0 \\
&\mathbf{n} \bullet \Delta\mathbf{I} + \nabla \bullet \mathbf{J} = 0 \\
&\mathbf{n} \times \Delta\mathbf{J} \bullet \mathbf{t} ds = 0
\end{aligned}$$

Considering $\mathbf{I}$ and $\mathbf{J}$ as representing currents, the first equation expresses that the amount which flows out of a small region is equal to the amount which flows in. The second equation expresses that the total flow from a portion of a conducting surface into space and along the surface is zero. The third equation expresses that the flow from one sub-region across a curve on a conducting surface is equal to the flow into the adjacent sub-region. These three equations thus express that $\mathbf{I}$ and $\mathbf{J}$, considered as space and surface currents, represent a flow of something which is conserved. For $\mathbf{I}$ and $\mathbf{J}$ to have this property the above discussion shows it is necessary and sufficient that A be everywhere solenoidal. In hydrodynamics, the field $\mathbf{I}$ corresponds to vorticity and it clearly is solenoidal because it is the curl of velocity.

Vector and scalar potential

The previous section showed that a vector field can be determined from the divergence and curl of the vector field and the values on surfaces of discontinuities and bounding surfaces. The integral equations are useful for developing analytical solutions for simple systems. However, in hydrodynamics the vorticity is generally an unknown quantity. Thus it is useful to express the potentials as differential equations that are solved simultaneously for the potentials and vorticity. The differential equations are derived by substituting the potentials into the expressions for the divergence and curl.
Equation:

$$\nabla^2\phi = -4\pi\rho$$
$$\nabla^2\mathbf{A} = -4\pi\mathbf{I}$$

In two-dimensional vector fields the vector potential $\mathbf{A}$ and the vector $\mathbf{I}$ has a nonzero component only in the third direction. In hydrodynamics the nonzero component of the vector potential is the stream function $4\pi\mathbf{I}$ corresponds to the vorticity, which has only one nonzero component.

Assignment 3.1: Particle velocity and acceleration
Suppose a particle fixed on the surface of a steady-rotating sphere with radius, $\mathbf{R}$, has a constant speed, $|\nu| = v$.

a. Show that its acceleration is perpendicular to its velocity.
b. Show that the acceleration has a radial (from the center of the sphere) component $a_\rho = -v^2/R$.
c. Let the magnitude of the angular velocity be ω. Express the centrifugal acceleration (direction perpendicular to the axis of rotation), a in terms of R, ω and the angle of particle from the axis of rotation.

Assignment 3.2: Differential area and volume

a. Express the differential of area in term of the differential of the surface coordinates for a spherical surface using the spherical polar coordinate system.
b. Obtain the differential volume elements in cylindrical and spherical polars by the Jacobian and check with a simple geometrical picture.

Assignment 3.3: Differential operators

a. Derive the expression for the Laplacian of a scalar in Cartesian coordinates from the definition of the gradient and divergence.
b. Prove that: $\nabla \bullet (\phi \mathbf{a}) = \nabla\phi \bullet \mathbf{a} + \phi(\nabla \bullet \mathbf{a})$
c. Let $\mathbf{x}$ be the Cartesian coordinates of points in space and $r = |\mathbf{x}|$. Calculate the divergence and curl of $\mathbf{x}$ and the gradient and Laplacian of r and $1/r$. Note any singularities.
d. Prove the identities involving the curl operator.
e. Suppose a rigid body has the velocity field $\mathbf{v} = \mathbf{v}_{(t)} + \omega \times (\mathbf{x} - \mathbf{x}_o)$. Show that the curl of this velocity field is $\nabla \times \mathbf{v} = 2\omega$.

The Kinematics of Fluid Motion
Topics Covered in this Chapter

- Particle paths and material derivatives
- Streamlines
- Streaklines
- Dilatation
- Reynolds' transport theorem
- Conservation of mass and the equation of continuity
- Deformation and rate of strain
- Physical interpretation of the deformation tensor
- Principal axis of deformation
- Vorticity, vortex lines, and tubes

Reading assignment: Chapter 4 of Aris

Kinematics is the study of motion without regard to the forces that bring about the motion. Already, we have described how rigid body motion is described by its translation and rotation. Also, the divergence and curl of the field and values on boundaries can describe a vector field. Here we will consider the motion of a fluid as microscopic or macroscopic bodies that translate, rotate, and deform with time. We treat fluids as a continuum such that the fluid identified to be at a specific point in space at one time with neighboring fluid will be at another specific point in space at a later time with the same neighbors, with the exception of certain bifurcations. This identification of the fluid occupying a point in space requires that the motion is deterministic rather than stochastic, i.e., random motions such as diffusion and turbulence are not described. Central to the kinematics of fluid motion is the concept of convection or following the motion of a "particle" of fluid.

Particle paths and material derivatives

Fluid motion will be described as the motion of a "particle" that occupies a point in space. At some time, say $t = 0$, a fluid particle is at a position $\xi = (\xi_1, \xi_2, \xi_3)$ and at a later time the same particle is at a position $\mathbf{x}$. The motion of the particle that occupied this original position is described as follows.
Equation:

$$\mathbf{x} = \mathbf{x}(\xi, t) \text{ or } x_i = x(\xi_1, \xi_2, \xi_3, t)$$

The initial coordinates ξ of a particle will be referred to as the *material coordinates* of the particles and, when convenient, the particle itself may be called the particle ξ . The terms *convected* and *Lagrangian* coordinates are also used. The *spatial coordinates* $\mathbf{x}$ of the particle may be referred to as its *position* or *place*. It will be assumed that the motion is continuous, single valued and the previous equation can be inverted to give the initial position or material coordinates of the particle which is at any position $\mathbf{x}$ at time t; i.e.,
Equation:

$$\xi = \xi(\mathbf{x}, t) \text{ or } \xi_i = \xi_i(x_1, x_2, x_3, t)$$

are also continuous and single valued. Physically this means that a continuous arc of particles does not break up during the motion or that the particles in the neighborhood of a given particle continue in its neighborhood during the motion. The single valuedness of the equations mean that a particle cannot split up and occupy two places nor can two distinct particles occupy the same place. Exceptions to these requirements may be allowed on a finite number of singular surfaces, lines or points, as for example a fluid divides around an obstacle. It is shown in Appendix B that a necessary and sufficient condition for the inverse functions to exist is that the Jacobian
Equation:

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)}$$

should not vanish.

The transformation $\mathbf{x} = \mathbf{x}(\xi, t)$ may be looked at as the parametric equation of a curve in space with t as the parameter. The curve goes through the point ξ , corresponding to the parameter $t = 0$, and these curves are the *particle paths*. Any property of the fluid may be followed along the particle path. For example, we may be given the density in the neighborhood of a particle as a function $\rho(\xi, t)$, meaning that for any prescribed particle ξ we have the density as a function of time, that is, the density that an observer riding on the particle would see. (Position itself is a "property" in this general sense so that the equations of the particle path are of this form.) This *material* description of the change of some property, say $\mathfrak{I}(\xi, t)$, can be changed to a spatial description $\mathfrak{I}(\mathbf{x}, \mathbf{t})$.
Equation:

$$\mathfrak{I}(\mathbf{x}, t) = \mathfrak{I}[\xi(\mathbf{x}, t), t]$$

Physically this says that the value of the property at position $\mathbf{x}$ at time t is the value appropriate to the particle that is at $\mathbf{x}$ at time t. Conversely, the material description can be derived from the spatial one.
Equation:

$$\mathfrak{I}(\xi, t) = \mathfrak{I}[\xi(\mathbf{x}, t), t]$$

meaning that the value as seen by the particle at time t is the value at the position it occupies at that time.
Equation:

$$\frac{\partial}{\partial t} \equiv \left(\frac{\partial}{\partial t}\right)_x = \text{ derivative with respect to time keeping } \mathbf{x} \text{ constant}$$

and
Equation:

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_{\xi} = \text{ derivative with respect to time keeping } \xi \text{ constant}$$

Thus $\partial \Im / \partial t$ is the rate of change of $\Im$ as observed at a fixed point $\mathbf{x}$, whereas $D\Im/Dt$ is the rate of changed as observed when moving with the particle, i.e., for a fixed value of ξ . The latter we call the *material derivative*. It is also called the convected, convective, or substantial derivative and often denoted by D/Dt. In particular the material derivative of the position of a particle is it velocity. Thus putting $\Im = x_i$, we have
Equation:

$$v_i = \frac{Dx_i}{Dt} = \frac{\partial}{\partial t} x_i \left(\xi_1, \xi_2, \xi_3, t\right)$$

or
Equation:

$$\mathbf{v} = \frac{D\mathbf{x}}{Dt}$$

This allows us to establish a connection between the two derivatives, for
Equation:

$$\begin{aligned} \frac{D\Im}{Dt} &= \frac{\partial}{\partial t} \Im\left(\xi, t\right) = \frac{\partial}{\partial t} \Im\left[x\left(\xi, t\right), t\right] \\ &= \left(\frac{\partial \Im}{\partial t}\right)_x + \frac{\partial \Im}{\partial t_i} \left(\frac{\partial x_i}{\partial t}\right)_{\xi} \\ &= \frac{\partial \Im}{\partial t} + v_i \frac{\partial \Im}{\partial x_i} \\ &= \frac{\partial \Im}{\partial t} + (\mathbf{v} \bullet \nabla) \Im \end{aligned}$$

Streamlines

We now have a formal definition of the velocity field as a material derivative of the position of a particle.
Equation:

$$\mathbf{v}\left(\mathbf{x}, t\right) = \frac{D\mathbf{x}(\xi, t)}{Dt} = \frac{\partial \mathbf{x}(\xi, t)}{\partial t}$$

The field line lines of the velocity field are called *streamlines*; they are the solutions of the three simultaneous equations

Equation:

$$\frac{d\mathbf{x}}{ds} = \mathbf{v}\left(\mathbf{x}, t\right)$$

where s is a parameter along the streamline. This parameter s is not to be confused with the time, for in the above equation t is held fixed while the equations are integrated, and the resulting curves are the streamlines *at the instant*t. These may vary from instant to instant and in general will not coincide with the particle paths.

To obtain the *particle paths* from the velocity field we have to follow the motion of each particle. This means that we have to solve the differential equations
Equation:

$$\frac{D\mathbf{x}}{Dt} = \mathbf{v}\left(\mathbf{x}, t\right)$$

subject to $\mathbf{x} = \xi$ at $t = 0$. Time is the parameter along the particle path. Thus the particle path is the trajectory taken by a particle.

The flow is called *steady* if the velocity components are independent of time. For steady flows, the parameter s along the streamlines may be taken to be t and the streamlines and particle paths will coincide. The converse does not follow as there are unsteady flows for which the streamlines and particle paths coincide.

If C is a closed curve in the region of flow, the streamlines through every point of C generate a surface known as a *stream tube*. Let S be a surface with C as the bounding curve, then
Equation:

$$\int\int_{s} \mathbf{v} \bullet \mathbf{n} dS$$

is known as the *strength* of the *stream* tube at its cross-section S.

The *acceleration* or the rate of change of velocity is defined as
Equation:

$$\begin{aligned} \mathbf{a} &= \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla)\mathbf{v} \\ a_i &= \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \end{aligned}$$

Notice that in steady flow this does not vanish but reduces to
Equation:

$$\mathbf{a} = (\mathbf{v} \bullet \nabla)\mathbf{v} \text{ for steady flow.}$$

Even in steady flow other than a constant translation, a fluid particle will accelerate if it changes direction to go around an obstacle or if it increases its speed to pass through a constriction.

Streaklines

The name streakline is applied to the curve traced out by a plume of smoke or dye, which is continuously injected at a fixed point but does not diffuse. Thus at time t the streakline through a fixed-point $\mathbf{y}$ is a curve going from $\mathbf{y}$ to $\mathbf{x}(\mathbf{y}, t)$, the position reached by the particle which was at $\mathbf{y}$ at time $t = 0$. A particle is on the streakline if it passed the fixed-point $\mathbf{y}$ at some time between 0 and t. If this time was t', then the material coordinates of the particle would be given by $\xi = \xi(\mathbf{y}, t')$. However, at time t this particle is at $\mathbf{x} = \mathbf{x}(\xi, t)$ so that the equation of the streakline at time t is given by
Equation:

$$\mathbf{x} = \mathbf{x}\big[\xi\big(\mathbf{y}, t'\big), t\big],$$

where the parameter t' along it lies in the interval $0 \le t' \le t$. If we regard the motion as having been proceeding for all time, then the origin of time is arbitrary and t' can take negative values $-\infty \le t' \le t$.

The flow field illustrated in 4.13 by Aris is assigned as an exercise.

Dilatation

We noticed earlier that if the coordinate system is changed from coordinates ξ to coordinates x, then the element of volume changes by the formula
Equation:

$$dV = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} d\xi_1 \, d\xi_2 \, d\xi_3 = J \, dV_0$$

If we think of ξ as the material coordinates, they are the Cartesian coordinates at $t = 0$, so that $d\xi_1 d\xi_2 d\xi_3$ is the volume dV_o of an elementary rectangular parallelepiped. Consider this elementary parallelepiped about a given point ξ at the initial instant. By the motion this parallelepiped is moved and distorted but because the motion is continuous it cannot break up and so at some time t is some neighborhood of the point $\mathbf{x} = \mathbf{x}(\xi, t)$. By the above equation, its volume is $dV = J \, dV_o$ and hence
Equation:

$$J = \frac{dV}{dV_o} = \text{ ratio of an elementary material volume to its initial volume.}$$

It is called the *dilation* or *expansion*. The assumption that $\mathbf{x} = \mathbf{x}(\xi, t)$ can be inverted to give $\xi = \xi(\mathbf{x}, t)$, and vice versa, is equivalent to requiring that neither J nor $J^{-}1$ vanish. Thus,
Equation:

$$0 \leq J \leq \infty$$

We can now ask how the dilation changes as we follow the motion. To answer this we calculate the material derivative DJ/Dt. However,
Equation:

$$J = \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix}$$

Now
Equation:

$$\frac{D}{Dt}\left(\frac{\partial x_i}{\partial \xi_j}\right) = \frac{\partial}{\partial \xi_j} \frac{Dx_i}{Dt} = \frac{\partial v_i}{\partial \xi_j}$$

for D/Dt is differentiation with ξ constant so that the order can be interchanged. Now if we regard v_i as a function of x_1, x_2, x_3,
Equation:

$$\frac{\partial v_i}{\partial \xi_j} = \frac{\partial v_i}{\partial \xi_j} \frac{\partial x_m}{\partial \xi_j}$$

The above relation can now be applied to differentiation of the Jacobian.
Equation:

$$\begin{aligned} \frac{DJ}{Dt} &= \varepsilon_{ijk} \frac{D}{Dt}\left(\frac{\partial x_i}{\partial \xi_1}\right) \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{D}{Dt}\left(\frac{\partial x_j}{\partial \xi_2}\right) \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{D}{Dt}\left(\frac{\partial x_k}{\partial \xi_3}\right) \\ &= \varepsilon_{ijk} \frac{\partial v_i}{\partial x_m} \frac{\partial x_m}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial v_j}{\partial x_m} \frac{\partial x_m}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial v_k}{\partial x_m} \frac{\partial x_m}{\partial \xi_3} \\ &= \varepsilon_{ijk} \frac{\partial v_1}{\partial x_1} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial v_i}{\partial x_2} \frac{\partial x_j}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3} + \varepsilon_{ijk} \frac{\partial x_i}{\partial \xi_1} \frac{\partial x_j}{\partial \xi_2} \frac{\partial v_3}{\partial x_3} \\ &= \frac{\partial v_1}{\partial x_1} J + \frac{\partial v_2}{\partial x_2} J + \frac{\partial v_3}{\partial x_3} J \\ &= (\nabla \bullet \mathbf{v}) J \end{aligned}$$

where we made use of the property of the determinant that the determinant of a matrix with repeated rows is zero. Thus,
Equation:

$$\frac{D(\ln J)}{Dt} = \nabla \bullet \mathbf{v}$$

We thus have an important physical meaning for the divergence of the velocity field. It is the relative rate of dilation following a particle path. It is evident that for an incompressible fluid motion,
Equation:

$$\nabla \bullet \mathbf{v} = 0 \text{ for incompressible fluid motion.}$$

Use of a stream function to satisfy the mass-conservation equation (Batchelor, 1967)

In the cases of flow of an incompressible fluid, and of steady flow of a compressible fluid, the mass-conservation equation reduces to the statement that a vector divergence is zero, the divergences being of $\mathbf{u}$ and $\rho\mathbf{u}$ respectively. If we impose the further restriction that the flow field either is two-dimensional or has axial symmetry, this vector divergence is the sum of only two derivatives, and the mass-conservation equation can then be regarded as defining a scalar function from which the components of $\mathbf{u}$ or $\rho\mathbf{u}$ are obtained by differentiation. The procedure will be described here for the case of an incompressible fluid.

Assume first that the motion is two-dimensional, so that $\mathbf{u} = (u, v, 0)$ and u and v are independent of z. The mass-conservation equation for an incompressible fluid then has the form
Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

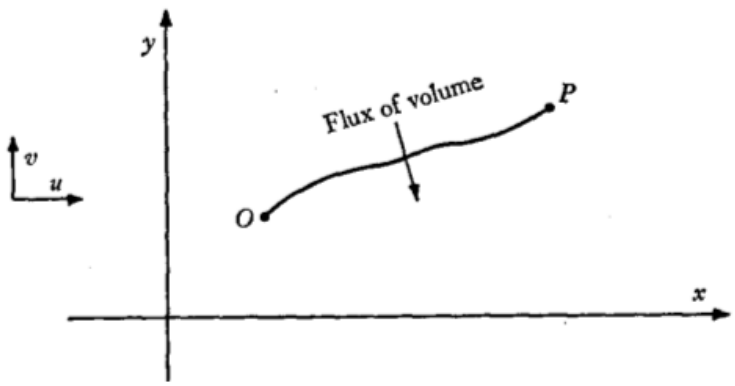
from which it follows that $u\delta y - v\delta x$ is an exact differential, equal to $\delta\psi$ say. Then
Equation:

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x},$$

and the unknown scalar function $\psi(x, y, t)$ is defined by
Equation:

$$\psi - \psi_o = \int (u\ dy - v\ dx),$$

where ψ_o is a constant and the line integral is taken along an arbitrary curve joining some reference point O to the point P with co-ordinates x, y. In this way we have used the mass-conservation equation to replace the two dependent variables u, v by the single dependent variable ψ , which is a very valuable simplification in many cases of two-dimensional flow.



Calculation of the flux of fluid volume across a curve joining the reference point O to the point P with co-ordinates (x, y)

The physical content of the above argument also proves to be of interest. The flux of fluid volume across a curve joining the points 0 and P in the (x, y)-plane (by which is meant the flux across the open surface swept out by translating this curve through unit distance in the z-direction), the flux being reckoned positive when it is in the anti-clockwise sense about P, is given exactly by the right-hand side of (2.2.8) (see figure 2.2.1). Now the flux of volume across the closed curve formed from any two different paths joining 0 to P is necessarily zero when the region between the two paths is wholly occupied by incompressible flow. The flux represented by the integral in (2.2.8) is therefore independent of the choice of the path joining O to P, provided it is one of a set of paths of which any two enclose only incompressible fluid, and therefore defines a function of the position of P, which we have written as $\psi - \psi_o$.

Since the flux of volume across any curve joining two points is equal to the difference between the values of ψ at these two points, it follows that ψ is constant along a streamline, as is also apparent from (2.2.7) and the equation (2.1.1) that defines the streamlines. ψ is termed the *stream function*, and is associated (in this case of two-dimensional flow) with the name of Lagrange. The function can also be regarded as the only non-zero component of a 'vector potential' for $\mathbf{u}$ (analogous to the vector potential of the magnetic induction, which also is a solenoidal vector, in electromagnetic field theory), since (2.2.7) can be written as

Equation:

$$\mathbf{u} = \nabla \times \mathbf{A}, \ \mathbf{A} = (0, 0, \psi)$$

It is common practice in fluid mechanics to provide a picture of a flow field by drawing various streamlines, and if these lines are chosen so that the two values of ψ on every pair of neighboring streamlines differ by the same amount, ε say, the eye is able to perceive the way in which the velocity magnitude q, as well as its direction, varies over the field, since
Equation:

$$q \approx \varepsilon//(\text{distance between neighboring streamlines}).$$

Examples of families of streamlines describing two-dimensional flow fields, with equal intervals in ψ between all pairs of neighboring streamlines, will be found in figures 2.6.2 and 2.7.2.

Expressions for the velocity components parallel to any orthogonal coordinate lines in terms of ψ may be obtained readily, either by the use of (2.2.9) or with the aid of the relation between ψ and the volume flux 'between two points'. For flow referred to polar co-ordinates (r,θ), we find, by evaluating the flux between pairs of neighboring points on the r- and θ-co-ordinate lines and equating it to the corresponding increments in ψ (allowance being made for the signs in the manner required by (2.2.8)), that
Equation:

$$u, = \frac{1}{r}\frac{\partial \psi}{\partial \theta},\ u_\theta = -\frac{\partial \psi}{\partial r}$$

The reader may find useful the general rule for two-dimensional flow, that differentiation of ψ in a certain direction gives the velocity component 90 degree in the clockwise sense from that direction.

Finally, for this case of two-dimensional flow of incompressible fluid, we should note the possibility that ψ is a many-valued function of position. For suppose that across some closed inner geometrical boundary there is a net volume flux m; this flux might be due to an effective creation of fluid within the inner boundary (as when a tube discharges fluid into this region) or to change of volume of the part of the enclosed region not occupied by the fluid (as when a gaseous cavity surrounded by water expands or contracts).

If now we choose two different paths joining the two points 0 and P which together make up a closed curve enclosing the inner boundary, the fluxes of volume across the two joining curves differ by an amount m (or, more generally, by pm, where p is the number of times the combined closed curve passes round the inner boundary). The value of $\psi - \psi_o$ at the point P thus depends on the choice of path joining it to the reference point 0, and may take any one of a number of values differing by multiples of m. This kind of many-valuedness of a scalar function related to the velocity distribution in a region which is not singly-connected will be described more fully in § 2.8. It is not confined to two-dimensional flow, although that is the context in which it occurs most often.

If now the flow has symmetry about an axis, the mass-conservation equation for an incompressible fluid takes the form
Equation:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{1}{\sigma}\frac{\partial(\sigma v)}{\partial \sigma} = 0$$

in terms of cylindrical co-ordinates (x, σ, φ) with corresponding velocity components (u, v, w), the axis of symmetry being the line $\sigma = 0$. This relation ensures that $\sigma u \delta\sigma - \sigma v \delta x$ is an exact differential, equal to $\delta\psi$ say. Then
Equation:

$$u = \frac{1}{\sigma}\frac{\partial \psi}{\partial \sigma},\ v = -\frac{1}{\sigma}\frac{\partial \psi}{\partial x}$$

and the function $\psi(x, \sigma, t)$ is defined by
Equation:

$$\psi - \psi_o = \int \sigma\,(u d\sigma - v dx),$$

where the line integral is taken along an arbitrary curve in an axial plane joining some reference point 0 to the point P with co-ordinates (x, σ). It will be noticed that the azimuthal component of velocity w does not enter into the mass-conservation equation in a flow field with axial symmetry and cannot be obtained from ψ.

Again it is possible to interpret ψ both as a measure of volume flux and as one component of a vector potential. The flux of fluid volume across the surface formed by rotating an arbitrary curve joining 0 to P in an axial plane, about the axis of symmetry, the flux again being reckoned as positive when it is in the anti-clockwise sense about P, is 2π-times the right-hand side of (2.2.12). Lines in an axial plane on which ψ is constant are everywhere parallel to the vector (u, v , o), and can be described as `streamlines of the flow in an axial plane'. ψ is here termed the *Stokes stream function*. A sketch of lines on which ψ is constant, with the same increment in ψ between all pairs of neighboring lines (see figure 2.5.2 for an example), does not give quite as direct an impression of the distribution of velocity magnitude here as in two-dimensional flow, owing to the occurrence of the factor $1/\sigma$ in the expressions for u and v in (2.2.11). The relations (2.2. 11) are readily seen to be equivalent to
Equation:

$$\mathbf{u} = \nabla \times \mathbf{A},\ A_\varphi = \psi/\sigma$$

the components of the vector potential $\mathbf{A}$ referred to cylindrical co-ordinate lines here being independent of the azimuthal angle φ.

The relation between and the volume flux 'between two points' may be used to obtain expressions for the velocity components referred to other orthogonal systems of co-ordinates in terms of ψ. For instance, for flow with axial symmetry referred to spherical polar co-ordinates (r, θ, φ), we find, by evaluating the flux between pairs of neighboring points on the r- and θ -coordinate lines

and equating it to 2π times the corresponding increments in ψ (allowance being made for the signs in the manner required by (2.2.12)), that
Equation:

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

With this co-ordinate system, the vector potential for the velocity has the azimuthal component
Equation:

$$A_\varphi = \frac{\psi}{r\sin\theta}$$

Exercise

At time to the position of a material element of fluid has Cartesian coordinates (a, b, c) and the density of the fluid is ρ_o. At a subsequent time t the position coordinates and density of the element are (X, Y, Z) and ρ . Show that with this Lagrangian specification of the flow field the equation of mass conservation is
Equation:

$$\frac{\partial(X,Y,Z)}{\partial(a,b,c)} = \frac{\rho_o}{\rho}$$

Reynolds' transport theorem

An important kinematical theorem can be derived from the expression for the material derivative of the Jacobian. It is due to Reynolds and concerns the rate of change not of an infinitesimal element of volume but any volume integral. Let $\Im(\mathbf{x},t)$ be any function and $V(t)$ be a closed volume moving with the fluid, that is consisting of the same fluid particles. Then
Equation:

$$F(t) = \iiint\limits_{v(t)} \Im(\mathbf{x},t)dV$$

is a function of t that can be calculated. We are interested in its material derivative DF/Dt. Now the integral is over the varying volume $V(t)$ so we cannot take the differentiation through the integral sign. If, however, the integration were with respect to a volume in ξ-space it would be possible to interchange differentiation and integration since D/Dt is differentiation with respect to t keeping ξ constant. The transformation $\mathbf{x} = \mathbf{x}(\xi,t)$, $dV = JdV_o$ allows us to do just this, for $V(t)$ has been defined as a moving material volume and so come from some fixed volume V_o at $t = 0$. Thus

Equation:

$$\begin{aligned}
\frac{d}{dt}\iiint\limits_{v(t)} \Im\left(\mathbf{x},t\right)dV &= \frac{d}{dt}\iiint\limits_{v_o} \Im\left[\mathbf{x}\left(\xi,t\right),t\right]J\ dV_o \\
&= \iiint\limits_{v_o}\left(\frac{D\Im}{Dt}J+\Im\frac{DJ}{Dt}\right)dV_o \\
&= \iiint\limits_{v_o}\left(\frac{D\Im}{Dt}+\Im\left(\nabla\bullet\mathbf{v}\right)\right)J\ dV_o \\
&= \iiint\limits_{v(t)}\left(\frac{D\Im}{Dt}+\Im\left(\nabla\bullet\mathbf{v}\right)\right)dV
\end{aligned}$$

Since $D/Dt=\left(\partial/\partial t\right)+\mathbf{v}\bullet\nabla$ we can express this formula into a number of different forms. Substituting for the material derivative and collecting the gradient terms gives
Equation:

$$\begin{aligned}
\frac{d}{dt}\iiint\limits_{v(t)} \Im\left(\mathbf{x},t\right)dV &= \iiint\limits_{v(t)}\left(\frac{\partial\Im}{\partial t}+\mathbf{v}\bullet\nabla\Im+\Im\left(\nabla\bullet\mathbf{v}\right)\right)dV \\
&= \iiint\limits_{v(t)}\left(\frac{\partial\Im}{\partial t}+\nabla\bullet\left(\Im\mathbf{v}\right)\right)dV
\end{aligned}$$

Now applying Green's theorem to the second integral we have
Equation:

$$\frac{d}{dt}\iiint\limits_{v(t)} \Im\left(\mathbf{x},t\right)dV = \iiint v(t)\frac{\partial\Im}{\partial t}dV+\oint\limits_{S(t)}\left(\Im\mathbf{v}\right)\bullet\mathbf{n}dS$$

where $S(t)$ is the bounding surface of $V(t)$. This admits of an immediate physical picture for it says that the rate of change of the integral of $\Im$ within the moving volume is the integral of the rate of change of $\Im$ at a point plus the net flow of over the bounding surface. $\Im$ can be any scalar or tensor component, so that this is a kinematical result of wide application. It is going to be the basis for the conservation of mass, momentum, energy, and species. This approach to the conservation equations differs from the approach taken by Bird, Stewart, and Lightfoot. They perform a balance on a fixed volume of space and explicitly account for the convective flux across the boundaries.

Conservation of mass and the equation of continuity

Although the idea of mass is not a kinematical one, it is convenient to introduce it here and to obtain the continuity equation. Let $\rho(\mathbf{x}, t)$ be the mass per unit volume of a homogeneous fluid at position $\mathbf{x}$ and time t. Then the mass of any finite material volume $V(t)$ is
Equation:

$$m = \iiint\limits_{v(t)} \rho\left(\mathbf{x}, t\right) dV.$$

If V is a material volume, that is, if it is composed of the same particles, and there are no sources or sinks in the medium we take it as a principle that the mass does not change. By inserting $\mathfrak{I} = \rho$ in Reynolds' transport theorem we have
Equation:

$$\begin{aligned} \frac{dm}{dt} &= \iiint\limits_{v(t)} \left\{ \frac{D\rho}{Dt} + \rho\left(\nabla \bullet \mathbf{v}\right) \right\} dV \\ &= \iiint\limits_{v(t)} \left\{ \frac{\partial \rho}{\partial t} + \nabla \bullet \left(\rho\ \mathbf{v}\right) \right\} dV \\ &= 0 \end{aligned}$$

This is true for an arbitrary material volume and hence the integrand itself must vanish everywhere. It follows that
Equation:

$$\frac{D\rho}{Dt} + \rho\left(\nabla \bullet \mathbf{v}\right) = \frac{\partial \rho}{\partial t} + \nabla \bullet \left(\rho \mathbf{v}\right) = 0$$

which is the equation of continuity.

A fluid for which the density ρ is constant is called *incompressible*. In this case the equation of continuity becomes
Equation:

$$\nabla \bullet \mathbf{v} = 0 \ \text{incompressible flow}$$

and the motion is isochoric or the velocity field solenoidal.

Combining the equation of continuity with Reynolds' transport theorem for a function $\mathfrak{I} = \rho F$ we have
Equation:

$$
\begin{aligned}
\frac{d}{dt}\iiint\limits_{v(t)} \rho F dV &= \iiint\limits_{v(t)} \left\{ \frac{D}{Dt}(\rho F) + \rho F\left(\nabla \bullet \mathbf{v}\right) \right\} dV \\
&= \iiint\limits_{v(t)} \left\{ \rho \frac{DF}{Dt} + F\left(\frac{D\rho}{Dt} + \rho \nabla \bullet \mathbf{v} \right) \right\} dV \\
&= \iiint\limits_{v(t)} \rho \frac{DF}{Dt} dV
\end{aligned}
$$

This equation is useful for deriving the conservation equation of a quantity that is expressed as specific to a unit of mass, e.g., specific internal energy and species mass fraction.

Deformation and rate of strain

The motion of fluids differs from that of rigid bodies in the deformation or strain that occurs with motion. Material coordinates give a reference frame for this deformation or strain.

Consider two nearby points P and Q with material coordinates ξ and $\xi + d\xi$. At time t they are to be found at $\mathbf{x}(\xi, t)$ and $\mathbf{x}(\xi + d\xi, t)$. Now
Equation:

$$
x_i(\xi + d\xi,\, t) = x_i(\xi,\, t) + \frac{\partial x_i}{\partial \xi_j}\, d\xi_j + O(d^2)
$$

where $O\left(d^2\right)$ represents terms of order $d\xi^2$ and higher which will be neglected from this point onward. Thus the small displacement vector $d\xi$ has now become
Equation:

$$
d\mathbf{x} = \mathbf{x}\left(\xi + d\xi, t\right) - \mathbf{x}\left(\xi, t\right)
$$

where
Equation:

$$
dx_i = \frac{\partial x_i}{\partial \xi_j} d\xi_j.
$$

It is clear from the quotient rule (since $d\xi$ is arbitrary) that the nine quantities $\partial x_i / \partial \xi_j$ are the components of a tensor. It may be called the displacement gradient tensor and is basic to the theories of elasticity and rheology. For fluid motion, its material derivative is of more direct application and we will concentrate on this.

If $\mathbf{v} = D\mathbf{x}/Dt$ is the velocity, the relative velocity of two particles ξ and $\xi + d\xi$ has components
Equation:

$$dv_i = \frac{\partial v_i}{\partial \xi_k} d\xi_k = \frac{D}{Dt}\left(\frac{\partial x_i}{\partial \xi_k}\right) d\xi_k$$

However, by inverting the above relation, we have
Equation:

$$dv_i = \frac{\partial v_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j}\, dx_j = \frac{\partial v_i}{\partial x_j}\, dx_j$$

expressing the relative velocity in terms of current position. Again it is evident that the $(\partial v_i / \partial x_j)$ are components of a tensor, the *velocity gradient tensor*, for which we need to obtain a sound physical feeling.

We first observe that if the motion is a rigid body translation with a constant velocity $\mathbf{u}$,
Equation:

$$\mathbf{x} = \xi + \mathbf{u}t$$

and the velocity gradient tensor vanishes identically. Secondly, the velocity gradient tensor can be written as the sum of symmetric and antisymmetric parts,
Equation:

$$\begin{aligned} \frac{\partial v_i}{\partial x_j} &\equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \\ &= e_{ij} + \Omega_{ij} \\ &\text{or} \\ \nabla \mathbf{v} &= \mathbf{e} + \Omega \end{aligned}$$

We have seen that a relative velocity dv_i related to the relative position dx_j by an antisymmetric tensor Ω_{ij}, i.e., $dv_i = \Omega_{ij}\, dx_j$, represents a rigid body rotation with angular velocity $\boldsymbol{\omega} = -vec\boldsymbol{\Omega}$. In this case
Equation:

$$\begin{aligned} \omega_i &= -\frac{1}{2}\varepsilon_{ijk}\Omega_{jk} = \frac{1}{2}\varepsilon_{ijk}\frac{\partial v_k}{\partial x_j} \\ &\text{or} \\ \omega &= \frac{1}{2}\nabla \times \mathbf{v} \end{aligned}$$

Thus the antisymmetric part of the velocity gradient tensor corresponds to rigid body rotation, and, if the motion is a rigid one (composed of a translation plus a rotation), the symmetric part of the velocity gradient tensor will vanish. For this reason the tensor e_{ij} is called the *deformation* or

rate of strain tensor and its vanishing is necessary and sufficient for the motion to be without deformation, that is, rigid.

Physical interpretation of the (rate of) deformation tensor

The (rate of) deformation tensor is what distinguishes fluid motion from rigid body motion. Recall that a rigid body is one in which the relative distance between two points in the body does not change. We show here that the (rate of) deformation tensor describes the rate of change of the relative distance between two particles in a fluid. Also, it describes the rate of change of the angle between three particles in the fluid.

First we will see how the distance between two material points change during the motion. The length of an infinitesimal line segment from P to Q is ds, where
Equation:

$$dx^2 = dx_i dx_i = \frac{\partial x_i}{\partial \xi_j}\frac{\partial x_i}{\partial \xi_k} d\xi_j d\xi_k$$

now P and Q are the material particles ξ and $\xi + d\xi$ so that $d\xi_j$ and $d\xi_k$ do not change during the motion. Also, recall that $Dx_i/Dt = v_i$. Thus
Equation:

$$\frac{D}{Dt}\left(ds^2\right) = \left(\frac{\partial v_i}{\partial \xi_j}\frac{\partial x_i}{\partial \xi_k} + \frac{\partial x_i}{\partial \xi_k}\frac{\partial v_i}{\partial \xi_k}\right) d\xi_j d\xi_k = 2\frac{\partial v_i}{\partial \xi_j}\frac{\partial x_i}{\partial \xi_k} d\xi_j d\xi_k$$

by symmetry. However,
Equation:

$$\begin{aligned} \frac{\partial v_i}{\partial \xi_j} d\xi_j &= dv_i = \frac{\partial v_i}{\partial x_j} dx_j \text{ and } \frac{\partial x_i}{\partial \xi_k} d\xi_k = dx, \\ \text{since } \mathbf{v} &= \mathbf{v}[\mathbf{x}(\xi)] \text{ and } \mathbf{x} = \mathbf{x}(\xi) \end{aligned}$$

Thus
Equation:

$$\frac{1}{2}\frac{D}{Dt}\left(ds^2\right) = (ds)\frac{D}{Dt}(ds) = \frac{\partial v_i}{\partial x_j} dx_j dx_i = (e_{ij} + \Omega_{ij}) dx_j dx_i = e_{ij} dx_j dx_i$$

by symmetry, or
Equation:

$$\frac{1}{(ds)}\frac{D}{Dt}(ds) = e_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds}.$$

Now dx_i/ds is the *ith* component of a unit vector in the direction of the segment PQ, so that this equation says that the rate of change of the length of the segment as a fraction of its length is related to its direction through the deformation tensor.

In particular, if PQ is parallel to the coordinate axis 01 we have $d\mathbf{x}/ds = \mathbf{e}_{(1)}$ and
Equation:

$$\frac{1}{dx_1}\frac{D}{Dt}(dx_1) = e_{11} \text{ in direction of } 01$$

Thus e_{11} is the rate of longitudinal strain of an element parallel to the 01 axis. Similar interpretations apply to e_{22} and e_{33}.

Now let's examine the angle between two line segments during the motion. Consider the segment, PQ and PR where PQ is the segment $\xi + \mathbf{d}\xi$ as before and PR is the segment $\xi + \mathbf{d}\xi'$. If θ is the angle between them and ds' is the length of PR, we have from the scalar product,
Equation:

$$ds\ ds' \cos\theta = dx_i dx_i'$$

Differentiating with respect to time we have
Equation:

$$\begin{aligned}
\frac{D}{Dt}\left[ds\ ds' \cos\theta\right] &= \frac{D(dx_i dx_i')}{Dt} \\
&= dv_i dx_i' + dx_i dv_i' \\
&= \frac{\partial v_i}{\partial x_j} dx_j dx_i' + dx_i \frac{\partial v_i}{\partial x_j} dx_j' \\
&= \frac{\partial v_i}{\partial x_j} dx_j dx_i' + dx_j \frac{\partial v_j}{\partial x_i} dx_i' \\
&= \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx_j dx_i' \\
&= 2e_{ij} dx_j dx_i'
\end{aligned}$$

since $dv_i' = (\partial v_i/\partial x_j)dx_j'$. The i and j are dummy suffixes so we may interchange them in the first term on the right, then performing differentiation we have after dividing by $dsds'$
Equation:

$$
\begin{aligned}
\frac{1}{dsds'}\frac{D}{Dt}\left[dsds'\cos\theta\right] &= \cos\theta\left\{\frac{1}{ds}\frac{D}{Dt}ds+\frac{1}{ds'}\frac{D}{Dt}ds'\right\}-\sin\theta\frac{D\theta}{Dt} \\
&= \left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right)\frac{dx_j}{ds}\frac{dx'_i}{ds'} \\
&= 2e_{ij}\frac{dx'_i}{ds'}\frac{dx_j}{ds}
\end{aligned}
$$

Now suppose that dx' is parallel to the axis 01 and dx to the axis 02, so that (dx'_i/ds') and $(dx_j/ds)=\delta_{j2}$ and $\theta_{12}=\pi/2$. Then
Equation:

$$
-\frac{d\theta_{12}}{dt}=2e_{12}
$$

Thus e_{12} is to be interpreted as one-half the rate of decrease of the angle between two segments originally parallel to the 01 and 02 axes respectively. Similar interpretations are appropriate to e_{23} and e_{31}.

The fact that the rate of deformation tensor is linear in the velocity field has an important consequence. Since we may superimpose two velocity fields to form a third, it follows that the deformation tensor of this is the sum, of the deformation tensors of the fields from which it was superimposed. If $v_i=\lambda(i)x_i$(no summation on i) , we have a deformation which is the superposition of three stretching parallel to the three axis. However, if $v_1=f(x_2)$, $v_2=0$, $v_3=0$ so that only nonzero component of the deformation tensor is $e_12=½f'(x_2)$, the motion is one of pure shear in which elements parallel to the coordinate axis is not stretched at all. Note however that in pure stretching an element not parallel or perpendicular to the direction of stretching will suffer rotation. Likewise in pure shear an element not normal to or in the plane of shear will suffer stretching.

Principal axis of deformation

The rate of deformation tensor is a symmetric tensor and the principal axis of deformation can be found. They correspond to the eigenvalues of the matrix and the eigenvalues are the principal rates of strain. A set of particles that is originally on the surface of a sphere will be deformed to an ellipsoid whose axes are coincident with the principal axis.

Vorticity, vortex lines, and tubes

We have frequently reminders of rotating bodies of fluid such as tropical storms, hurricanes, tornadoes, dust devils, whirlpools, eddies in the flow behind objects, turbulence, and the vortex in draining bathtubs. The kinematics of these fluid motions is described by the vorticity.

The antisymmetric part of the rate of strain tensor Ω_{ij} represents the local rotation, ω_k. Recall
Equation:

$$
\begin{aligned}
\omega_i &= -\frac{1}{2}\varepsilon_{ijk}\Omega_{jk} \\
\omega &= -\mathrm{vec}\Omega \\
&= \frac{1}{2}\nabla \times \mathbf{v}
\end{aligned}
$$

The curl of velocity is known as the *vorticity*,
Equation:

$$
\mathbf{w} = \nabla \times \mathbf{v}
$$

Thus the vorticity and the antisymmetric part of the rate of strain tensor is a measure of the rotation of the velocity field. An irrotational flow field is one in which the vorticity vanishes everywhere. The field lines of the vorticity field are called vortex *lines* and the surface generated by the vortex lines through a closed curve C is a *vortex tube*. The strength of a vortex tube is defined as the surface integral of the normal component. It is equal to the circulation around the closed curve C that bounds the cross-section S by Stokes' theorem.
Equation:

$$
\begin{aligned}
\iint_s \mathbf{w} \bullet \mathbf{n} dS &= \iint_s (\nabla \times \mathbf{v}) \bullet \mathbf{n} dS \\
&= \oint \mathbf{v} \bullet \mathbf{t} ds \\
&= \Gamma
\end{aligned}
$$

We observe that the strength of a vortex tube at any cross-section is the same, as $\mathbf{w}$ is a solenoidal vector. The surface integral of the normal component of a solenoidal vector vanishes over any closed surface. The surface integral on the surface of a vortex tube is zero because the sides are tangent to the vorticity vector field. Thus the surface integral across any cross-section must be equal in magnitude. The magnitude of the vorticity field can be visualized from the relative width of the vortex tubes in the same manner that the magnitude of the velocity field can be visualized by the width of the stream tubes.

Because the strength of the tube does not vary with position along the tube, it follows that the vortex tubes are either closed, go to infinity or end on solid boundaries of rotating objects. In a real fluid satisfying the no-slip boundary condition, vortex lines must be tangential to the surface of a body at rest, except at isolated points of attachment and separation, because the normal component of vorticity vanishes on the stationary solid.

When the vortex tube is immediately surrounded by irrotational fluid, it will be referred to as a vortex *filament*. A vortex filament is often just called a vortex, but we shall use this term to denote any finite volume of vorticity immersed in irrotational fluid. Of course, the vortex filament and the vortex require the fluid to be ideal (zero viscosity) to make strict sense, because viscosity diffuses vorticity, but they are useful approximations for real fluids of small viscosity.

Helmholtz gave three laws of vortex motion in 1858. For the motion of an ideal (zero viscosity) barotropic (density is a single valued function of pressure) fluid under the action of conservative external body forces (gradient of a scalar), they can be expressed as follows:

I. Fluid particles originally free of vorticity remain free of vorticity.
II. Fluid particles on a vortex line at any instant will be on a vortex line at subsequent times. Alternatively, it can be said that vortex lines and tubes move with the fluid.
III. The strength of a vortex tube does not vary with time during the motion of the fluid.

The equations for the dynamics of vorticity will be developed later.

P. G. Saffman, Vortex Dynamics, Cambridge University Press, 1992. C. Truesdell, The Kinematics of Vorticity, Indiana University Press, 1954.

Assignment 4.1
Plot the streamlines, particle paths, and streaklines of the flow field described in Sec. 4.13. Find the CHBE 501 web page. Download the files in CHBE501/Problems/lines. Execute *lines* with MATLAB.

Assignment 4.2
Execute the program *deform* in CHBE 501/Problems/deform and print the figures. It computes the particle paths for a patch of particles deforming in Couette flow, stagnation flow, and that of Sec 4.13. Look at the respective subroutine to determine the equations of the flow field. For each of these flow fields calculate:

a. divergence
b. curl
c. rate of deformation tensor
d. antisymmetric tensor
e. Which fields are solenoidal or irrotational?

Stress in Fluids

Topics Covered in this Chapter

- Cauchy's stress principle and the conservation of momentum
- The stress tensor
- The symmetry of the stress tensor
- Hydrostatic pressure
- Principal axes of stress and the notion of isotropy
- The Stokesian fluid
- Constitutive equations of the Stokesian fluid
- The Newtonian fluid
- Interpretation of the constants λ and μ

Reading assignment

Chapter 1 in BSL
Chapter 5 in Aris

The only material property of the fluid we have so far discussed is the density. In the last chapter we introduced the rate of deformation or rate of strain tensor. The distinguishing characteristic between fluids and solids is that fluids can undergo unlimited deformation and yet maintain its integrity. The relation between the rate of deformation tensor and stress tensor is the *mechanical constitutive equation* of the material. An ideal fluid has a stress tensor that is independent of the rate of deformation, i.e., it has an isotropic component, which is identified as the pressure and has zero viscosity.

Cauchy's stress principle and the conservation of momentum

The forces acting on an element of a continuous medium may be of two kinds. *External* or *body forces*, such as gravitation or electromagnetic forces, can be regarded as reaching into the medium and acting throughout the volume. If the external force can be describes as the gradient of a scalar, the force is said to be conservative. *Internal* or *contact forces* are to be regarded as acting on an element of volume through its bounding surfaces. If an element of volume has an external-bounding surface, the forces there may be specified, e.g., when a constant pressure is applied over a free surface. If the element is internal, the resultant force is that exerted by the

material outside the surface upon that inside. Let $\mathbf{n}$ be the unit outward normal at a point of the surface S and $\mathbf{t}_{(n)}$ the force per unit area exerted there by the material outside S. Then Cauchy's principle asserts that $\mathbf{t}_{(n)}$ is a function of the position $\mathbf{x}$, the time t, and the orientation $\mathbf{n}$ of the surface element. Thus the total internal force exerted on the volume V through the bounding surface S is

Equation:

$$\iint_s \mathbf{t}_{(n)} dS.$$

If $\mathbf{f}$ is the external force per unit mass (e.g. if 03 is vertical, gravitation will exert a force $-g\mathbf{e}_{(3)}$ per unit mass or $-\rho g\mathbf{e}_{(3)}$ per unit volume, the total external force will be

Equation:

$$\iiint_v \rho \mathbf{f} dV.$$

The principle of the conservation of linear momentum asserts that the sum of these two forces equals the rate of change of linear momentum of the volume, i.e.,

Equation:

$$\frac{D}{Dt} \iiint_v \rho \mathbf{v} dV = \iiint_v \rho \mathbf{f} dV + \iint_s \mathbf{t}_{(n)} dS$$

This is just a generalization of Newton's law of motion, which states that the rate of change of momentum of a particle is equal to the sum of forces acting on it. It has been extended to a volume that contains a number of particles.

From the form of these integral relations we can deduce an important relation. Suppose V is a volume of a given shape with characteristic dimension d. Then the volume of V will be proportional to d^3 and the area of S to d^2, with the proportionality constants depending only on the shape. Now let V shrink to a point but preserve its shape, then the volume integrals in the last equation will decrease as d^3 but the surface integral will decrease as d^2. It follows that

Equation:

$$\lim_{d \to 0} \frac{1}{d^2} \iint_s \mathbf{t}_{(n)} dS = 0$$

or, *the stresses are locally in equilibrium*.

The stress tensor

To elucidate the nature of the stress system at a point P we consider a small tetrahedron with three of its faces parallel to the coordinate planes through P and the fourth with normal n (see Fig. 5.1 of Aris). If dA is the area of the slanted face, the areas of the faces perpendicular to the coordinate axis *Pi* is

Equation:

$$dA_i = n_i dA.$$

The outward normals to these faces are $-\mathbf{e}_{(i)}$ and we may denote the stress vector over these faces by $-\mathbf{t}_{(i)}$. ($\mathbf{t}_{(i)}$ denotes the stress vector when $+\mathbf{e}_{(i)}$ is the outward normal.) Then applying the principle of local equilibrium to the stress forces when the tetrahedron is very small we have

Equation:

$$\begin{aligned} &\mathbf{t}_{(n)} dA - \mathbf{t}_{(1)} dA_1 - \mathbf{t}_{(2)} dA_2 - \mathbf{t}_{(3)} dA_3 \\ &= \big(\mathbf{t}_{(n)} - \mathbf{t}_{(1)} n_1 - \mathbf{t}_{(2)} n_2 \mathbf{t}_{(3)} n_3\big) dA = 0 \end{aligned}$$

Now let T_{ji} denote the ith component of $\mathbf{t}_{(j)}$ and $t_{(n)i}$ the ith component of $\mathbf{t}_{(n)}$ so that this equation can be written
Equation:

$$t_{(n)i} = T_{ji} n_j.$$

However, $\mathbf{t}_{(n)}$ is a vector and $\mathbf{n}$ is a unit vector quite independent of the T_{ji} so that by the quotient rule the T_{ji} are components of a second order tensor $\mathbf{T}$. In dyadic notation we might write
Equation:

$$\mathbf{t}_{(n)} = \mathbf{T} \bullet \mathbf{n}.$$

This tells us that the system of stresses in a fluid is not so complicated as to demand a whole table of functions $\mathbf{t}_{(n)}(\mathbf{x}, \mathbf{n})$ at any given instant, but that it depends rather simply on $\mathbf{n}$ through the nine quantities $T_{ji}(\mathbf{x})$. Moreover, because these are components of a tensor, any equation we derive with them will be true under any rotation of the coordinate axis.

Inserting the tensor expression for the stress into the momentum balance and using the equation of continuity and Green's theorem we have
Equation:

$$
\begin{aligned}
\frac{D}{Dt}\iiint\limits_v \rho\mathbf{v}dV &= \iiint\limits_v \rho\mathbf{f}dV + \iint\limits_s \mathbf{t}_{(n)}dS \\
&= \iiint\limits_v \rho\mathbf{f}dV + \iint\limits_s \mathbf{n}\bullet\mathbf{T}dS \\
&= \iiint\limits_v (\rho\mathbf{f} + \nabla\bullet\mathbf{T}dV \\
\frac{D}{Dt}\iiint\limits_v \rho\mathbf{v}dV &= \iiint\limits_v \left[\frac{D}{Dt}(\rho\mathbf{v}) + \rho\mathbf{v}\left(\nabla\bullet\mathbf{v}\right)\right]dV \\
&= \iiint\limits_v \left[\rho\frac{D\mathbf{v}}{Dt} + \mathbf{v}\left(\frac{D\rho}{Dt} + \rho\nabla\bullet\mathbf{v}\right)\right]dV \\
&= \iiint\limits_v \rho\frac{D\mathbf{v}}{Dt}dV
\end{aligned}
$$

Since all the integrals are now volume integrals, they can be combined as a single integrand.

Equation:

$$
\iiint\limits_v \left(\rho\frac{d\mathbf{v}}{Dt} - \rho\mathbf{f} - \nabla\bullet\mathbf{T}\right)dV = 0
$$

However, since V is an arbitrary volume this equation is satisfied only if the integrand vanishes identically.

Equation:

$$
\begin{aligned}
\rho\frac{D\mathbf{v}}{Dt} &= \rho\mathbf{f} + \nabla \bullet \mathbf{T} \\
&= \rho\mathbf{a} \\
&\text{or} \\
\rho\frac{Dv_i}{Dt} &= \rho f_i + T_{ji,j} \\
&= \rho\alpha_i
\end{aligned}
$$

This is Cauchy's equation of motion and $\mathbf{a}$ is the acceleration. It holds for any continuum no matter how the stress tensor $\mathbf{T}$ is connected with the rate of strain.

The symmetry of the stress tensor

A polar fluid is one that is capable of transmitting stress couples and being subject to body torques, as in magnetic fluids. In case of a polar fluid we must introduce a body torque per unit mass in addition to the body force and a couple stress in addition to the normal stress $\mathbf{t}_{(n)}$. The stress for polar fluids is discussed by Aris.

A fluid is nonpolar if the torques within it arise only as the moments of direct forces. For the nonpolar fluid we can make the assumption either that angular momentum is conserved or that the stress tensor is symmetric. We will make the first assumption and deduce the symmetry.

Return now to the integral linear momentum balance with the internal force expressed as a surface integral. If we assume that all torques arise from macroscopic forces, then not only linear momentum but also the angular momentum $\mathbf{x} \times (\rho\mathbf{v})$ are expressible in terms of $\mathbf{f}$ and $\mathbf{t}_{(n)}$.

Equation:

$$\iiint\limits_v \rho \mathbf{a} dV = \iiint\limits_v \rho \mathbf{f} dV + \iint\limits_s \mathbf{t}_{(n)} dS$$

$$\iiint\limits_v \rho \left(\mathbf{x} \times \mathbf{a}\right) dV = \iiint\limits_v \rho \left(\mathbf{x} \times \mathbf{f}\right) dV + \iint\limits_s (\mathbf{x} \times \mathbf{t}_{(n)} dS$$

The surface integral has as its ith component
Equation:

$$\begin{aligned} \iint\limits_s (\mathbf{x} \times \mathbf{t}_{(n)} dS &= \varepsilon_{ijk} x_j T_{kp} n_p dS \\ &= \iiint\limits_v \varepsilon_{ijk} \left(x_j \mathbf{T}_{pk}\right)_{,p} dV \end{aligned}$$

by Green's theorem. However, since $x_{j,p} = \delta_{jp}$, this last integrand is
Equation:

$$\begin{aligned} \varepsilon_{ijk} \left(x_j T_{pk}\right)_{,p} &= \varepsilon_{ijk} x_j T_{pk,p} + \varepsilon_{ijk} T_{jk} \\ &\text{or} \\ &= \mathbf{x} \times \left(\nabla \bullet \mathbf{T}\right) + \mathbf{T}_{\times} \end{aligned}$$

where $\mathbf{T}$ is the vector $\varepsilon_{ijk}\ \mathbf{T}_{jk}$.

Since $\mathbf{v} \times \mathbf{v} = 0,\ d(\mathbf{x} \times \mathbf{v})/dt = \mathbf{x} \times \mathbf{a}$, applying the transport theorem to the angular momentum we have
Equation:

$$\frac{D}{Dt} \iiint\limits_v \rho \left(\mathbf{x} \times \mathbf{v}\right) dV = \iiint\limits_v \rho \left(\mathbf{x} \times \mathbf{a}\right) dV$$

Substituting back into the equation for the angular momentum and rearranging gives
Equation:

$$\iiint_V \mathbf{v} \times (\rho \mathbf{a} - \rho \mathbf{f} - \nabla \bullet \mathbf{T}) dV = \iiint_V \mathbf{T}_\times dV$$

However, the left-hand side vanishes for an arbitrary volume and so
Equation:

$$T_\times \equiv 0.$$

The components of $\mathbf{T}_\times$ are $(T_{23} - T_{32})$, $(T_{31} - T_{13})$, and $(T_{12} - T_{21})$ and the vanishing of these implies
Equation:

$$T_{ij} = T_{ji}$$

so that $\mathbf{T}$ is symmetric for nonpolar fluids.

Hydrostatic pressure

If the stress system is such that an element of area always experiences a stress normal to itself and this stress is independent of orientation, the stress is called *hydrostatic*. All fluids at rest exhibit this stress behavior. It implies that $\mathbf{n} \bullet \mathbf{T}$ is always proportional to $\mathbf{n}$ and that the constant of proportionality is independent of $\mathbf{n}$. Let us write this constant $-p$, then
Equation:

$$\begin{aligned} n_i T_{ij} &= -p n_j, \text{ hydrostatic stress} \\ \mathbf{n} \bullet \mathbf{T} &= p \mathbf{n} \end{aligned}$$

However, this equation means that any vector is a characteristic vector or eigenvector of $\mathbf{T}$. This implies that the hydrostatic stress tensor is spherical or isotropic. Thus
Equation:

$$\begin{aligned} T_{ij} &= -p\delta_{ij}, \text{ hydrostatic stress} \\ \mathbf{T} &= -p\mathbf{I} \end{aligned}$$

for the state of hydrostatic stress.

For a compressible fluid at rest, p may be identified with the classical thermodynamic pressure. On the assumption that there is local thermodynamic equilibrium even when the fluid is in motion this concept of stress may be retained. For an incompressible fluid the thermodynamic, or more correctly thermostatic, pressure cannot be defined except as the limit of pressure in a sequence of compressible fluids. We shall see later that it has to be taken as an independent dynamical variable.

The stress tensor for a fluid may always be written
Equation:

$$\begin{aligned} T_{ij} &= p\delta_{ij} + P_{ij} \\ \mathbf{T} &= p\mathbf{I} + \mathbf{P} \end{aligned}$$

and P_{ij} is called the *viscous stress tensor*. The viscous stress tensor of a fluid vanishes under hydrostatic conditions.

If the external or body force is conservative (i.e., gradient of a scalar) the hydrostatic pressure is determined up to an arbitrary constant from the potential of the body force.
Equation:

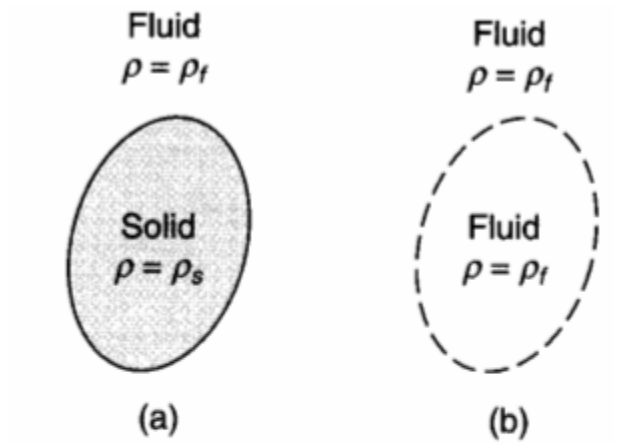
$$
\begin{aligned}
\nabla \bullet \mathbf{T} &= -\rho \mathbf{f}, \text{ static conditions} \\
\mathbf{f} = \mathbf{g} &= g \nabla z, \text{ uniform gravity field} \\
\nabla \bullet \mathbf{T} &= -\nabla p, \text{ hydrostatic conditions} \\
\nabla p = \rho \nabla \Phi & \\
\Phi(p) &= \int^{p} \frac{dp}{\rho} + C_1 = g\, z + C_2 \\
p = \rho\, g\, z + C_3, \text{ if } \rho \text{ is constant} &
\end{aligned}
$$

Buoyancy (Deen, 1998)

A consequence of the fact that pressure increases with depth in a static fluid is that the pressure exerts a net upward force on any submerged object. To calculate this force, consider an object of arbitrary shape submerged in a constant-density fluid as shown in the figure. The net force on this object, F_p, is given by

Equation:

$$
\mathbf{F}_p = -\iint_s p \mathbf{n} dS
$$



Pressure forces on a submerged object. The control volume in (b) is used to calculate the pressure force on the object in (a).

where the minus sign reflects the fact that positive pressure are compressive (i.e., pressure acts in the $-\mathbf{n}$ direction). The pressure force is evaluated most easily by considering the situation in the figure where the solid has been replaced by an identical volume of fluid. Because the pressure in the fluid depends on depth only, this replacement of the solid does not affect the pressure distribution in the surrounding fluid. In particular, the pressure distribution on the fluid control surface in (b) must be identical to that on the solid surface in (a). Situation (b) has the advantage that we can apply the divergence theorem to the integral in the last equation, because p is a continuous function of position within the volume V; this is not necessarily true for (a), because we have said nothing about the meaning of p within a solid. Applying the divergence theorem and using the hydrostatic pressure field, we find that

Equation:

$$
\begin{aligned}
\mathbf{F}_p &= -\iint_s p\mathbf{n}dS \\
&= -\iiint_v \nabla p dV = -\rho_f \mathbf{g} V
\end{aligned}
$$

where ρ_f is the (constant) density of the fluid. Thus, the upward force on the object due to pressure equals the weight of an equivalent volume of liquid; this is **Archimedes' law**. An important implication of this equation is that F_p is independent of the absolute pressure (provided that the density is independent of pressure).

The **buoyancy force** is the net force on the object due to the same static pressure variation and gravity. Evaluating the gravitational force on the solid body by setting $\rho = \rho_f$, the buoyancy force is found to be
Equation:

$$\mathbf{F}_B = (\rho_i - \rho_f)\mathbf{g}V.$$

Principal axes of stress and the notion of isotropy

The diagonal terms T_{11}, T_{22}, T_{33} of the stress tensor are sometimes called the *direct stresses* and the terms T_{12}, T_{21}, T_{31}, T_{13}, T_{23}, T_{32} the *shear stresses*. When there are no external or stress couples, the stress tensor is symmetric and we can invoke the known properties of symmetric tensors. In particular, there are three principal directions and referred to coordinates parallel to these, the shear stresses vanish. The direct stresses with these coordinates are called the *principal stresses* and the axes the *principal axes of stress*.

An *isotropic* fluid is such that simple direct stress acting in it does not produce a shearing deformation. This is an entirely reasonable view to take for isotropy means that there is no internal sense of direction within the fluid. Another way of expressing the absence of any internally preferred direction is to say that the functional relation between stress and rate of

deformation must be independent of the orientation of the coordinate system. We shall show in the next section that this implies that the principal axes of stress and rate of deformation coincide.

The Stokesian fluid

The constitutive equation of a non-elastic fluid satisfying the hypothesis of Stokes is called a Stokesian fluid. This fluid is based on the following assumptions.

I. The stress tensor T_{ij} is a continuous function of the rate of deformation tensor e_{ij} and the local thermodynamic state, but independent of other kinematical quantities.
II. The fluid is homogeneous, that is, T_{ij} does not depend explicitly on $\mathbf{x}$.
III. The fluid is isotropic, that is, there is no preferred direction.
IV. When there is no deformation $(e_{ij} = 0)$ the stress is hydrostatic, $(T_{ij} = -p\delta_{ij})$.

The first assumption implies that the relation between the stress and rate of strain is independent of the rigid body rotation of an element given by the antisymmetric kinematical tensor Ω_{ij}. The thermodynamic variables, for example, pressure and temperature, will be carried along this discussion without specific mention except where it is necessary for emphasis. We are concerned with a homogeneous portion of fluid so the second assumption is that the stress tensor depends only on position through the variation of e_{ij} and thermodynamic variables with position. The third assumption is that of isotropy and this implies that the principal directions of the two tensors coincide. To express this as an equation we write $T_{ij} = f_{ij}(e_{pq})$, then if there is no preferred direction T_{ij} is the same function f_{ij} of e_{pq} as T_{ij} is of e_{pq}. Thus
Equation:

$$T_{ij} = f_{ij}(e_{pq}).$$

The fourth assumption is that the tensor

Equation:

$$P_{ij} = T_{ij} + p\delta_{ij}$$

vanishes when there is no motion. P_{ij} is called the viscous stress tensor.

Constitutive equations of the Stokesian fluid

The arguments for the form of the constitutive equation for the Stokesian fluid is given by Aris and is not repeated here. The equation takes the form
Equation:

$$\begin{aligned} &T_{ij} = -p\delta_{ij} + \beta e_{ij} + \gamma e_{ik} e_{kj}, \\ &\mathbf{T} = -p\mathbf{I} + \beta \mathbf{e} + \gamma \mathbf{e} \bullet \mathbf{e} \end{aligned}$$

which insures that T_{ij} reduces to the hydrostatic form when the rate of deformation vanishes.

The Newtonian fluid

The Newtonian fluid is a linear Stokesian fluid, that is, the stress components depend linearly on the rates of deformation. Aris gives two arguments that deduce the form of the constitutive equation for a Newtonian fluid.
Equation:

$$\begin{aligned} &T_{ij} = (-p + \lambda\Theta)\delta_{ij} + 2\mu e_{ij} \\ &\text{where} \\ &\Theta = e_{ij} = \nu_{i,i} = \nabla \bullet \mathbf{v} \end{aligned}$$

Interpretation of the constants λ and μ

Consider the shear flow given by

Equation:

$$v_1 = f(x_2), \nearrow v_2 = v_3 = 0$$

For this we have all the e_{ij} zero except
Equation:

$$e_{12} = e_{21} = \frac{1}{2} f'(x_2) = \frac{1}{2} \frac{\partial v_1}{\partial x_2}$$

Thus
Equation:

$$P_{12} = P_{21} = \mu f'(x_2) = \mu \frac{\partial v_1}{\partial x_2}$$

and all other viscous stresses are zero. It is evident that μ is the proportionality constant relating the shear stress to the velocity gradient. This is the common definition of the viscosity, or more precisely the coefficient of shear viscosity of a fluid.

For an incompressible, Newtonian fluid the pressure is the mean of the principal stresses since this is
Equation:

$$\begin{aligned} \frac{1}{3} T_{ii} &= -p + \lambda \Theta + \frac{2}{3} \mu \Theta \\ &= -p \end{aligned}$$

For a compressible fluid we should take the pressure p as the thermodynamic pressure to be consistent with our ideas of equilibrium. Thus if we call $-p$ the mean of the principal stresses,
Equation:

$$
\begin{aligned}
p - p &= -\left(\lambda + \frac{2}{3}\mu\right)\Theta \\
&= -\left(\lambda + \frac{2}{3}\mu\right)\nabla \bullet \mathbf{v} \\
&= -\left(\lambda + \frac{2}{3}\mu\right)\frac{1}{\rho}\frac{D\rho}{Dt}
\end{aligned}
$$

Since p, the thermodynamic pressure, is in principle known from the equation of state $p - p$ is a measurable quantity. The coefficient in the equation is known as the *coefficient of bulk viscosity*. It is difficult to measure, however, since relatively large rates of change of density must be used and the assumption of linearity is then dubious. Stokes assumed that $p = \underline{p}$ and on this ground claimed that

Equation:

$$
\lambda + \frac{2}{3}\mu = 0
$$

supporting this from an argument from the kinetic theory of gases.

Assignment 5.1

Assume a Newtonian fluid and calculate the stress tensor for the flow fields of assignment 4.2. Evaluate the force due to the stress on the surface $y = 0$.

Equations of Motion and Energy in Cartesian Coordinates

Topics Covered in this Chapter

- Equations of motion of a Newtonian fluid
- The Reynolds number
- Dissipation of Energy by Viscous Forces
- The energy equation
- The effect of compressibility
- Resume of the development of the equations
- Special cases of the equations
    - Restrictions on types of motion
        - Isochoric motion
        - Irrotational motion
        - Plane flow
        - Axisymmetric flow
        - Parallel flow perpendicular to velocity gradient
    - Specialization on the equations of motion
        - Hydrostatics
        - Steady flow
        - Creeping flow
        - Inertial flow
        - Boundary layer flow
        - Lubrication and film flow
    - Specialization of the constitutive equation
        - Incompressible flow
        - Perfect (inviscid, nonconducting) fluid
        - Ideal gas
        - Piezotropic fluid and barotropic flow
        - Newtonian fluids
- Boundary conditions
    - Surfaces of symmetry
    - Periodic boundary
    - Solid surfaces
    - Fluid surfaces
    - Boundary conditions for the potentials and vorticity
- Scaling, dimensional analysis, and similarity
    - Dimensionless groups based on geometry
    - Dimensionless groups based on equations of motion and energy

    - Friction factor and drag coefficients

- Bernoulli theorems
    - Steady, barotropic flow of an inviscid, nonconducting fluid with conservative body forces
    - Coriolis force
    - Irrotational flow
    - Ideal gas

Reading assignment
Chapter 2&3 in BSL
Chapter 6 in Aris

Equations of motion of a Newtonian fluid

We will now substitute the constitutive equation for a Newtonian fluid into Cauchy's equation of motion to derive the Navier-Stokes equation.

Cauchy's equation of motion is
Equation:

$$\begin{aligned} \rho\alpha_i &= \rho\frac{Dv_i}{Dt} = \rho f_i + T_{ij,j} \\ \text{or} \\ \rho\mathbf{a} &= \rho\frac{D\mathbf{v}}{Dt} = \rho\mathbf{f} + \nabla \bullet \mathbf{T} \end{aligned}$$

The constitutive equation for a Newtonian fluid is
Equation:

$$\begin{aligned} T_{ij} &= (-p + \lambda\Theta)\delta_{ij} + 2\mu e_{ij} \\ \text{or} \\ \mathbf{T} &= (-p + \lambda\Theta)\mathbf{I} + 2\mu\mathbf{e} \end{aligned}$$

The divergence of the rate of deformation tensor needs to be restated with a more meaningful expression.
Equation:

$$
\begin{aligned}
e_{ij,j} &= \frac{1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right) \\
&= \frac{1}{2}\frac{\partial^2 v_i}{\partial x_j \partial x_j}+\frac{1}{2}\frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial x_j} \\
&= \frac{1}{2}\nabla^2 v_i+\frac{1}{2}\frac{\partial}{\partial x_i}(\nabla \bullet \mathbf{v}). \\
&\text{or} \\
\nabla \bullet \mathbf{e} &= \frac{1}{2}\nabla^2 \mathbf{v}+\frac{1}{2}\nabla(\nabla \bullet \mathbf{v})
\end{aligned}
$$

Thus
Equation:

$$
\begin{aligned}
&T_{ij,j} = -\frac{\partial p}{\partial x_i}+(\lambda+\mu)\frac{\partial}{\partial x_i}(\nabla \bullet \mathbf{v})+\mu\nabla^2 v_i \\
&\text{or} \\
&\rho\frac{D\mathbf{v}}{Dt} = \rho\mathbf{f}-\nabla p+(\lambda+\mu)\nabla\Theta+\mu\nabla^2\mathbf{v}
\end{aligned}
$$

Substituting this expression into Cauchy's equation gives the Navier-Stokes equation.
Equation:

$$
\begin{aligned}
&\rho\frac{Dv_i}{Dt} = \rho\, f_i-\frac{\partial p}{\partial x_i}+(\lambda+\mu)\frac{\partial}{\partial x_i}(\nabla \bullet \mathbf{v})+\mu\nabla^2 v_i \\
&\text{or} \\
&\rho\frac{D\mathbf{v}}{Dt} = \rho\mathbf{f}-\nabla p+(\lambda+\mu)\nabla\Theta+\mu\nabla^2\mathbf{v}
\end{aligned}
$$

The Navier-Stokes equation is sometimes expressed in terms of the acceleration by dividing the equation by the density.
Equation:

$$
\begin{aligned}
\mathbf{a} = \frac{D\mathbf{v}}{Dt} &= \frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v} \bullet \nabla)\mathbf{v} \\
&= \mathbf{f}-\frac{1}{\rho}\nabla p+\left(\lambda'+v\right)\nabla\Theta+v\nabla^2\mathbf{v}
\end{aligned}
$$

where $v = \mu/\rho$ and $\lambda' = \lambda/\rho$. ν is known as the kinematic viscosity and if Stokes' relation is assumed $\lambda'+\nu = \nu/3$. Using the identities
Equation:

$$\begin{aligned}\nabla^2\mathbf{v} &\equiv \nabla(\nabla\bullet\mathbf{v})-\nabla\times(\nabla\times\mathbf{v})\\ \mathbf{w} &\equiv \nabla\times\mathbf{v}\end{aligned}$$

the last equation can be modified to give
Equation:

$$\mathbf{a}=\frac{D\mathbf{v}}{Dt}=\mathbf{f}-\frac{1}{\rho}\nabla p+\left(\lambda'+2\nu\right)\nabla\Theta-\nu\nabla\times\mathbf{w}.$$

If the body force f can be expressed as the gradient of a potential (conservative body force) and density is a single valued function of pressure (piezotropic), the Navier-Stokes equation can be expressed as follows.
Equation:

$$\mathbf{a}=\frac{D\mathbf{v}}{Dt}=-\nabla\left[\Omega+P\left(p\right)-\left(\lambda'+2\nu\right)\Theta\right]-\nu\nabla\times\mathbf{w}$$
$$\text{where}$$
$$\mathbf{f}=-\nabla\Omega\ \text{and}\ P\left(p\right)=\int^{p}\frac{dp'}{\rho(p')}$$

Assignment 6.1
Do exercises 6.11.1, 6.11.2, 6.11.3, and 6.11.4 in Aris.

The Reynolds number

Later we will discuss the dimensionless groups resulting from the differential equations and boundary conditions. However, it is instructive to derive the Reynolds number N_{Re} from the Navier-Stokes equation at this point. The Reynolds number is the characteristic ratio of the inertial and viscous forces. When it is very large the inertial terms dominate the viscous terms and vice versa when it is very small. Its value gives the justification for assumptions of the limiting cases of inviscid flow and creeping flow.

We will consider the case of single-phase flow with conservative body forces (e.g., gravitational) and density a single valued function of pressure. The pressure and potential from the body force can be combined into a single potential.
Equation:

$$\mathbf{f}-\frac{1}{\rho}\nabla p=-\nabla\Omega$$
$$where$$
$$\Omega=\int^{p}\frac{dp}{\rho}-gz$$

If the change in density is small enough, the potential can be approximated by potential that has the units of pressure.
Equation:

$$\Omega \approx \frac{P}{\rho}, \text{ small change in density}$$
$$\text{where}$$
$$P = p - \rho g z$$

Suppose that the flow is characterized by a certain linear dimension, L, a velocity U, and a density ρ . For example, if we consider the steady flow past an obstacle, L may be it's diameter and U and ρ the velocity and density far from the obstacle. We can make the variables dimensionless with the following
Equation:

$$\begin{aligned} \mathbf{v}^* &= \frac{\mathbf{v}}{U}, \ \mathbf{x}^* \frac{\mathbf{x}}{L}, \ t^* = \frac{U}{L} t, \ P^* = \frac{P}{\rho U^2} \\ \nabla^* &= L\nabla, \ \nabla^{*2} = L^2 \nabla^2 \end{aligned}$$

The conservative body force, Navier-Stokes equation is made dimensionless with these variables.
Equation:

$$\begin{aligned} \rho \frac{D\mathbf{v}}{Dt} &= -\nabla P + (\lambda + \mu)\nabla\Theta + \mu \nabla^2 \mathbf{v} \\ \rho \frac{U^2}{L} \frac{D\mathbf{v}^*}{Dt^*} &= -\rho \frac{U^2}{L} \nabla^* P^* + \frac{\mu\, U}{L^2} (\lambda/\mu + 1) \nabla^* \Theta^* + \frac{\mu\, U}{L^2} \nabla^{*2} \mathbf{v}^* \\ \frac{\rho U L}{\mu} \left[\frac{D\mathbf{v}^*}{Dt^*} + \nabla^* P^* \right] &= (\lambda/\mu + 1) \nabla^* \Theta^* + \nabla^{*2} \mathbf{v}^* \\ N_{Re} \left[\frac{D\mathbf{v}^*}{Dt^*} + \nabla^* P^* \right] &= (\lambda/\mu + 1) \nabla^* \Theta^* + \nabla^{*2} \mathbf{v}^* \\ & \text{where} \\ N_{Re} &= \frac{\rho U\, L}{\mu} = \frac{\rho U^2}{\mu U / L} \end{aligned}$$

The Reynolds number partitions the Navier -Stokes equation into two parts. The left side or inertial and potential terms, which dominates for large NRe and the right side or viscous terms, which dominates for small NRe. The potential gradient term could have been on the right side if the dimensionless pressure was defined differently, i.e., normalized with respect to $(\mu U)/L$, the shear stress rather than kinetic energy. Note that the left side has only first derivatives of the spatial variables while the right side has second derivatives. We will see later that the left side

may dominate for flow far from solid objects but the right side becomes important in the vicinity of solid surfaces.

The nature of the flow field can also be seen form the definition of the Reynolds number. The second expression is the ratio of the characteristic kinetic energy and the shear stress.

The alternate form of the dimensionless Navier-Stokes equation with the other definition of dimensionless pressure is as follows.
Equation:

$$N_{Re}\frac{D\mathbf{V}^*}{Dt^*} = -\nabla^* P^{**} + (\lambda/\mu + 1)\nabla^*\Theta^* + \nabla^{*2}\mathbf{v}^*$$
$$P^{**} = \frac{P}{\mu U/L}$$

Dissipation of Energy by Viscous Forces

If there was no dissipation of mechanical energy during fluid motion then kinetic energy and potential energy can be exchanged but the change in the sum of kinetic and potential energy would be equal to the work done to the system. However, viscous effects result in irreversible conversion of mechanical energy to internal energy or heat. This is known as viscous dissipation of energy. We will identify the components of mechanical energy in a flowing system before embarking on a total energy balance.

The rate that work W is done on fluid in a material volume V with a surface S is the integral of the product of velocity and the force at the surface.
Equation:

$$\begin{aligned}\frac{dW}{dt} &= \oiint_s \mathbf{v} \bullet \mathbf{t}_{(n)}\, dS \\ &= \oiint_s \mathbf{v} \bullet \mathbf{T} \bullet \mathbf{n}\, dS \\ &= \iiint_v \nabla \bullet (\mathbf{v} \bullet \mathbf{T})\, dV\end{aligned}$$

The last integrand is rather complicated and is better treated with index notation.
Equation:

$$
\begin{aligned}
(v_i T_{ij})_{,j} &= T_{ij} v_{i,j} + v_i T_{ij,j} \\
&= T_{ij} v_{i,j} + v_i \left[\rho \frac{Dv_i}{Dt} - \rho f_i \right] \\
&= T_{ij} v_{i,j} + \frac{1}{2} \rho \frac{Dv_2}{Dt} - \rho f_i v_i \\
\nabla \bullet (\mathbf{v} \bullet \mathbf{t}) &= \mathbf{T} : \nabla \mathbf{v} + \frac{1}{2} \rho \frac{Dv^2}{Dt} - \rho \mathbf{f} \bullet \mathbf{v}
\end{aligned}
$$

We made use of Cauchy's equation of motion to substitute for the divergence of the stress tensor. The integrals can be rearranged as follows.
Equation:

$$
\begin{aligned}
\frac{d}{dt} \iiint_v \frac{1}{2} \rho v^2 dV &= \iiint_v \frac{1}{2} \frac{Dv^2}{Dt} dV \\
&= \iiint_v \rho \mathbf{f} \bullet \mathbf{v} dV + \iiint_v \nabla \bullet (\mathbf{v} \bullet \mathbf{T}) dV - \iiint_v \mathbf{T} : \nabla \mathbf{v} dV \\
\frac{1}{2} \rho \frac{Dv^2}{Dt} &= \rho \mathbf{f} \bullet \mathbf{v} + \nabla \bullet (\mathbf{v} \bullet \mathbf{T}) - \mathbf{T} : \nabla \mathbf{v} \\
&\text{where} \\
\mathbf{T} : \nabla \mathbf{v} &= T_{ij} v_{i,j}
\end{aligned}
$$

The left-hand term can be identified to be the rate of change of kinetic energy. The first term on the right-hand side is the rate of change of potential energy due to body forces. The second term is the rate at which surface stresses do work on the material volume. We will now focus attention on the last term.

The last term is the double contracted product of the stress tensor with the velocity gradient tensor. Recall that the stress tensor is symmetric for a nonpolar fluid and the velocity gradient tensor can be split into symmetric and antisymmetric parts. The double contract product of a symmetric tensor with an antisymmetric tensor is zero. Thus the last term can be expressed as a double contracted product of the stress tensor with the rate of deformation tensor. We will use the expression for the stress of a Newtonian fluid.
Equation:

$$
\begin{aligned}
-T_{ij} v_{i,j} &= -T_{ij} e_{ij} \\
&= -[(-p + \lambda \Theta) \delta_{ij} + 2\mu e_{ij}] e_{ij} \\
&= [p - \lambda \Theta] e_{ii} - 2\mu e_{ij} e_{ij} \\
&= p\Theta - \lambda \Theta^2 - 2\mu \left(\Theta^2 - 2\Phi \right) \\
&= -\mathbf{T} : \nabla \mathbf{v}
\end{aligned}
$$

where Φ is the second invariant of the rate of deformation tensor. Thus the rate at which kinetic energy per unit volume changes due to the internal stresses is divided into two parts:

i. a reversible interchange with strain energy, ,
ii. a dissipation by viscous forces,

Equation:

$$-\left[(\lambda+2\mu)\Theta^2-4\mu\Phi\right]$$

Since $\Theta^2-2\Phi$ is always positive, this last term is always dissipative. If Stokes' relation is used this term is
Equation:

$$-\mu\left[\frac{4}{3}\Theta^2-4\Phi\right]$$

for incompressible flow it is
Equation:

$$4\mu\Phi.$$

(The above equation is sometimes written $-4\mu\Phi$, where Φ is called the dissipation function. We have reserved the symbol Φ for the second invariant of the rate of deformation tensor, which however is proportional to the dissipation function for incompressible flow. Y is the symbol used later for the negative of the dissipation by viscous forces.)

The energy equation

We need the formulation of the energy equation since up to this point we have more unknowns than equations. In fact we have one continuity equation (involving the density and three velocity components), three equations of motion (involving in addition the pressure and another thermodynamic variable, say the temperature) giving four equations in six unknowns. We also have an equation of state, which in incompressible flow asserts that ρ is a constant reducing the number of unknowns to five. In the compressible case it is a relation
Equation:

$$\rho=f(p,T)$$

which increases the number of equations to five. In either case, there remains a gap of one equation, which is filled by the energy equation.

The equations of continuity and motion were derived respectively from the principles of conservation of mass and momentum. We now assert the first law of thermodynamics in the

form that the increase in total energy (we shall consider only kinetic and internal energies) in a material volume is the sum of the heat transferred and work done on the volume. Let $\mathbf{q}$ denote the heat flux vector, then, since $\mathbf{n}$ is the outward normal to the surface, $-\mathbf{q} \bullet \mathbf{n}$ is the heat flux into the volume. Let U denote the specific internal energy, then the balance expressed by the first law of thermodynamics is

Equation:

$$\frac{d}{dt}\iiint\limits_{v} \rho\left(\frac{v^2}{2}+U\right)dV = \iiint\limits_{v} \rho\mathbf{f} \bullet \mathbf{v}dV + \oiint\limits_{s} \mathbf{v} \bullet \mathbf{T} \bullet \mathbf{n}dS - \oiint\limits_{s} \mathbf{q} \bullet \mathbf{n}dS$$

This may be simplified by subtracting from it the expression we already have for the rate of change of kinetic energy, using Reynolds transport theorem, and Green's theorem.

Equation:

$$\iiint\limits_{v}\left[\rho\frac{DU}{Dt} + \nabla \bullet \mathbf{q} - \mathbf{T} : (\nabla\mathbf{v})\right]dV = 0$$

Since this is valid for any arbitrary material volume, we have assumed continuity of the integrand

Equation:

$$\rho\frac{DU}{Dt} = -\nabla \bullet \mathbf{q} + \mathbf{T} : (\nabla\mathbf{v}).$$

We assume Fourier's law for the conduction of heat.

Equation:

$$\mathbf{q} = -k\nabla T$$

We assume a Newtonian fluid for the dissipation of energy.

Equation:

$$\mathbf{T} : (\nabla\mathbf{v}) = -p(\nabla \bullet \mathbf{v}) + Y$$
$$Y = (\lambda + 2\mu)\Theta^2 - 4\mu\Phi$$

Substituting this back into the energy balance we have

Equation:

$$\rho\frac{DU}{Dt} = \nabla \bullet (k\,\nabla T) - p(\nabla \bullet \mathbf{v}) + \Upsilon$$

Physically we see that the internal energy increases with the influx of heat, the compression and the viscous dissipation.

If we write the equation in the form
Equation:

$$\rho \frac{DU}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = \nabla \bullet (k\nabla T) + Y$$

the left-hand side can be transformed by one of the fundamental thermodynamic identities. For if S is the specific entropy,
Equation:

$$\begin{aligned} T\,dS &= dU + pd(1/\rho) \\ &= dU - \left(p/\rho^2\right)d\rho \end{aligned}$$

Substituting this into the last equation for internal energy gives
Equation:

$$\rho T \frac{DS}{Dt} = \nabla \bullet (k\nabla T) + Y.$$

Giving an equation for the rate of change of entropy. Dividing by T and integrating over a material volume gives
Equation:

$$\begin{aligned} \iiint_v \rho \frac{DS}{Dt} dV &= \frac{d}{dt}\iiint_v \rho S dV \\ &= \iiint_v \left[\frac{1}{T}\nabla \bullet (k\nabla T) + \frac{Y}{T}\right] dV \\ &= \iiint_v \left[\nabla \bullet \left(\frac{k}{T}\nabla T\right) + \frac{k}{T^2}(\nabla T)^2 + \frac{Y}{T}\right] dV \\ &= \oiint_s \frac{-\mathbf{q} \bullet \mathbf{n}}{T} dS + \iiint_v \left[\frac{k}{T^2}(\nabla T)^2 + \frac{Y}{T}\right] dV \end{aligned}$$

The second law of thermodynamics requires that the rate of increase of entropy should be no less than the flux of heat divided by temperature. The above equation is consistent with this requirement because the volume integral on the right-hand side cannot be negative. It is zero only if k or ∇T and are zero. This equation also shows that entropy is conserved during flow if the thermal conductivity and viscosity are zero.

Equation:

$$\frac{DS}{Dt} = 0, \text{ when } \mu = 0, \text{ and } k = 0$$

Assignment 6.2
Do exercises 6.3.1 and 6.3.2 in Aris.

The Effect of Compressibility (Batcehlor, 1967)

Isentropic flow. The condition of zero viscosity and thermal conductivity results in conservation of entropy during flow or isentropic flow. This ideal condition is useful for illustration the effect of compressibility on fluid dynamics. The conservation of entropy during flow implies that density, pressure, and temperature are changing in a reversible manner during flow. The relation between entropy, density, temperature, and pressure is given by thermodynamics.
Equation:

$$\begin{aligned} dS &= \left(\frac{\partial S}{\partial T}\right)_p dT + \left(\frac{\partial S}{\partial p}\right)_T dp \\ &= \frac{C_p}{T} dT - \left(\frac{\partial 1/\rho}{\partial T}\right)_p dp \\ &= \frac{C_p}{T} dT - \frac{\beta}{\rho} dp \\ & \quad \text{where} \\ \beta &= -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \end{aligned}$$

These relations may be combined with the condition that the material derivative of entropy is zero to obtain a relation between temperature and pressure during flow.
Equation:

$$C_p \frac{DT}{Dt} = \frac{\beta T}{\rho} \frac{D\rho}{Dt}, \text{ when } \mu = 0, \text{ and } k = 0$$

The equation of state expresses the density as a function of temperature and pressure. During isentropic flow the pressure and temperature are not independent but are constrained by constant entropy or adiabatic compression and expansion. The density in this case is given as
Equation:

$$\rho = \rho(p, S)$$

We now have as many equations as unknowns and the system can be determined. The simplifying feature of isentropic flow is that exchanges between the internal energy and other forms of energy are reversible, and internal energy and temperature play passive roles, merely changing in response to the compression of a material element. The continuity equation and equation of motion governing isotropic flow may now be expressed as follows.

Equation:

$$\frac{1}{c^2}\frac{Dp}{Dt} + \rho\nabla \bullet \mathbf{v} = 0$$

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\mathbf{f} - \nabla p$$

$$\text{where}$$

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

The physical significance of the parameter c, which has the dimensions of velocity, may be seen in the following way. Suppose that a mass of fluid of uniform density ρ_o is initially at rest, in equilibrium, so that the pressure p_o is given by

Equation:

$$\nabla p_o = \rho_o \mathbf{f}.$$

The fluid is then disturbed slightly (all changes being isentropic), by some material being compressed and their density changed by small amounts, and is subsequently allowed to return freely to equilibrium and to oscillate about it. (The fluid is elastic, and so no energy is dissipated, so oscillations about the equilibrium are to be expected.) The perturbation quantities $\rho_1\ (= \rho - \rho_o)$ and $p_1\ (= p - p_o)$ and $\mathbf{v}$ are all small in magnitude and a consistent approximation to the continuity equation and equations of motion is

Equation:

$$\frac{1}{c_o^2}\frac{\partial p_1}{\partial t} + \rho_o\nabla \bullet \mathbf{v} = 0$$

$$\rho_o\frac{\partial \mathbf{v}}{\partial t} = \rho_1\mathbf{f} - \nabla p_1$$

where c_o is the value of c at $\rho = \rho_o$. On eliminating $\mathbf{v}$ we have

Equation:

$$\frac{1}{c_o^2}\frac{\partial^2 p_1}{\partial t^2} = \nabla^2 p_1 - \nabla \cdot (\rho_1 \mathbf{f})$$

The body forces commonly arise from the earth's gravitational field, in which case the divergence is zero and the last term is negligible except in the unlikely event of a length scale of the pressure variation not being small compared with c_o^2/g (which is about $1.2 \times 10^4 m$ for air under normal conditions and is even larger for water). Thus under these conditions the above equation reduces to the wave equation for p_1 and ρ_1 satisfies the same equation. The solutions of this equation represents plane compression waves, which propagate with velocity c_o and in which the fluid velocity $\mathbf{v}$ is parallel to the direction of propagation. In another words, c_o is the speed of propagation of sound waves in a fluid whose undisturbed density is ρ_o.

<u>Conditions for the velocity distribution to be approximately solenoidal.</u> The assumption of solenoidal or incompressible fluid flow is often made without a rigorous justification for the assumption. We will now reexamine this assumption and make use of the results of the previous section to express the conditions for solenoidal flow in terms of identifiable dimensionless groups.

The condition of solenoidal flow corresponds to the divergence of the velocity field vanishing everywhere. We need to characterize the flow field by a characteristic value of the change in velocity U with respect to both position and time and a characteristic length scale over which the velocity changes L. The spatial derivatives of the velocity then is of the order of U/L. The velocity distribution can be said to be approximately solenoidal if
Equation:

$$|\nabla \bullet \mathbf{v}| \ll \frac{U}{L}$$
$$\text{i.e., if}$$
$$\frac{1}{\rho}\frac{D\rho}{Dt} \ll \frac{U}{L}$$

For a homogeneous fluid we may choose ρ and the entropy per unit mass S as the two independent parameters of state, in which case the rate of change of pressure experienced by a material element can be expressed as
Equation:

$$p = p(\rho, S)$$
$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} \left(\frac{\partial p}{\partial S}\right)_\rho \frac{DS}{Dt}.$$

The condition that the velocity field should be approximately solenoidal is
Equation:

$$\frac{1}{\rho c^2}\frac{Dp}{Dt} - \frac{1}{\rho c^2}\left(\frac{\partial p}{\partial S}\right)_\rho \frac{DS}{Dt} \ll \frac{U}{L}.$$

This condition will normally be satisfied only if each of the two terms on the left-hand side has a magnitude small compared with U/L. We will now examine each of these terms.

I. When the condition
Equation:

$$\frac{1}{\rho c^2}\frac{Dp}{Dt} \ll \frac{U}{L}$$

is satisfied, the changes in density of a material element due to pressure variations are negligible, i.e., *the fluid is behaving as if it were incompressible*. This is by far the more practically important of the two requirements for $\mathbf{v}$ to be a solenoidal vector field. In estimating $|Dp/Dt|$ we shall lose little generality by assuming the flow to be isentropic, because the effects of viscosity and thermal conductivity are normally to modify the distribution of pressure rather than to control the magnitude of pressure variation. We may then rewrite the last equation with the aid of equations of motion of an isentropic fluid derived in the last section.
Equation:

$$\begin{aligned}\frac{Dp}{Dt} &= \frac{\partial p}{\partial t} + \mathbf{v} \bullet \nabla p \\ &= \frac{\partial p}{\partial t} + \mathbf{v} \bullet \left[\rho \mathbf{f} - \rho \frac{d\mathbf{v}}{dt}\right] \\ &= \frac{\partial p}{\partial t} + \rho \mathbf{v} \bullet \mathbf{f} - \frac{\rho}{2}\frac{dv^2}{dt}\end{aligned}$$

Thus
Equation:

$$\frac{1}{\rho c^2}\frac{\partial p}{\partial t} + \frac{\mathbf{v} \bullet \mathbf{f}}{c^2} - \frac{1}{2c^2}\frac{Dv^2}{Dt} \ll \frac{U}{L}.$$

Showing that in general three separate conditions, viz. that each term on the left-hand side should have a magnitude small compared with U/L if the flow field is to be incompressible.

- **I (i)** Consider first the last term on the left-hand side of the above equation. The order of magnitude of Dv^2/Dt will be the same as that of $\partial v^2 \partial/t$ or $\mathbf{v} \bullet \nabla v^2$ (i.e., U^3/L), which ever is greater. Thus the condition arising from this term can be expressed as
Equation:

$$\left(\frac{U}{c}\right)^2 \ll 1$$

or

$$N_{Ma} \ll 1$$

where

$$N_{Ma} = \frac{U}{c}$$

- **I (ii)** The magnitude of the partial derivative of pressure with respect to time depends directly on the unsteadiness of the flow. Let us suppose that the flow field is oscillatory and that ν is a measure of the dominant frequency. The rate of change of momentum is then the order of $\rho U \nu$. Since the pressure gradient is the order of the rate of change of momentum, the spatial pressure variation over a region of length L is $\rho L U \nu$. Since the pressure is also oscillating, the magnitude of $\partial p / \partial t$ is then $\partial L U \nu^2$. Thus the condition that the first term be small compared to U/L is
  Equation:

  $$\frac{\nu^2 L^2}{c^2} \ll 1.$$

  This condition is equivalent to the condition that the length of the system should be small enough that a pressure transient due to compression is felt instantaneously throughout the system.
- **I(iii)** If we regard the body forces arising from gravity, the term from the body forces, $\mathbf{v} \bullet \mathbf{f} / c^2$, has a magnitude of order gU/c^2, so the condition that it be small compared to U/L is
  Equation:

  $$\frac{v^2 L^2}{c^2} \ll 1$$

  This condition is equivalent to the condition that the length of the system should be small enough that a pressure transient due to compression is felt instantaneously throughout the system.

This shows that the condition is satisfied provided the difference between the static-fluid pressure at two points at vertical distance L apart is a small fraction of the absolute pressure, i.e., provided the length scale L characteristic of the velocity distribution is small compared to $p/\rho g$, the 'scale height' of the atmosphere, which is about 8.4 km for air under normal conditions. The fluid will thus behave as if it were incompressible when the three conditions I(i), I(ii), and I(iii) are satisfied. The first is not satisfied in near sonic or hypersonic gas dynamics, the second is not satisfied in acoustics, and the third is not satisfied dynamical meteorology.

II. The second condition necessary for incompressible flow is that arising from entropy. This condition requires that variation of density of a material element due to internal dissipative heating or due to molecular conduction of heat into the element be small. We will show

later how the small variation of density leading to natural convection can be allowed by yet assume incompressible flow.

Resume of the development of the equations

We have now obtained a sufficient number of equations to match the number of unknown quantities in the flow of a fluid. This does not mean that we can solve them nor even that the solution will exists, but it certainly a necessary beginning. It will be well to review the principles that have been used and the assumptions that have been made.

The foundation of the study of fluid motion lies in *kinematics*, the analysis of motion and deformations without reference to the forces that are brought into play. To this we added the concept of mass and the *principle of the conservation of mass, which leads to the equation of continuity,*
Equation:

$$\frac{D\rho}{Dt} + \rho\left(\nabla \bullet \mathbf{v}\right) = \frac{\partial \rho}{\partial t} + \nabla \bullet \left(\rho \mathbf{v}\right) = 0$$

An analysis of the nature of stress allows us to set up a stress tensor, which together with the *principle of conservation of linear momentum* gives the *equations of motion*
Equation:

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{f} + \nabla \bullet \mathbf{T}.$$

If the conservation of moment of momentum is assumed, it follows that the stress tensor is symmetric, but it is equally permissible to hypothesize the symmetry of the stress tensor and deduce the conservation moment of momentum. For a certain class of fluids however (hereafter called polar fluids) the stress tensor is not symmetric and there may be an internal angular momentum as well as the external moment of momentum.

As yet nothing has been said as to the constitution of the fluid and certain assumptions have to be made as to its behavior. In particular we have noticed that the hypothesis of Stokes that leads to the *constitutive equation* of a Stokesian fluid (not-elastic) and the linear Stokesian fluid which is the Newtonian fluid.
Equation:

$$\begin{aligned} T_{ij} &= (-p + \alpha)\delta_{ij} + \beta e_{ij} + \gamma e_{ik} e_{kj}, && \text{Stokesian fluid} \\ T_{ij} &= (-p + \lambda \Theta)\delta_{ij} + 2\mu e_{ij}, && \text{Newtonian fluid.} \end{aligned}$$

The coefficients in these equations are functions only of the invariants of the rate of deformation tensor and of the thermodynamic state. The latter may be specified by two thermodynamic variables and the nature of the fluid is involved in the equation of state, of which one form is

Equation:

$$\rho = f(p, T).$$

If we substitute the constitutive equation of a Newtonian fluid into the equations of motion, we have the Navier-Stokes equation.
Equation:

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{f} - \nabla p + (\lambda + \mu) \nabla (\nabla \bullet \mathbf{v}) + \mu \nabla^2 \mathbf{v}$$

Finally, the *principle of the conservation of energy* is used to give an energy equation. In this, certain assumptions have to be made as to the energy transfer and we have only considered the conduction of heat, giving
Equation:

$$\rho \frac{DU}{Dt} = \nabla \bullet (k \nabla T) - p (\nabla \bullet \mathbf{v}) + Y$$

These equations are both too general and too special. They are too general in the sense that they have to be simplified still further before any large body of results can emerge. They are too special in the sense that we have made some rather restrictive assumptions on the way, excluding for example elastic and electromagnetic effects.

Special cases of the equations

The full equations may be specialized is several ways, of which we shall consider the following:

i. restrictions on the type of motions,
ii. specializations on the equations of motion,
iii. specializations of the constitutive equation or equation of state.

This classification is not the only one and the classes will be seen to overlap. We shall give a selection of examples and of the resulting equations, but the list is by no means exhaustive.

Under the first heading we have any of the specializations of the velocity as a vector field. These are essentially kinematic restrictions.

- **(ia)** Isochoric motion. (i.e., constant density) The velocity field is solenoidal
  Equation:

$$-\frac{1}{\rho} \frac{D\rho}{Dt} = \nabla \bullet \mathbf{v} = \Theta = 0$$

The equation of continuity now gives
Equation:

$$\frac{D\rho}{Dt} = 0$$

that is, the density does not change following the motion. This does not mean that it is uniform, though, if the fluid is incompressible, the motion is isochoric. The other equations simplify in this case for we have α , β , and γ of the constitutive equations functions of only Φ and Ψ of the invariants of the rate of deformation tensor. In particular for a Newtonian fluid
Equation:

$$\begin{aligned} T_{ij} &= -p\delta_{ij} + 2\mu e_{ij} \\ \rho\frac{D\mathbf{v}}{Dt} &= \rho\mathbf{f} - \nabla p + \mu\nabla^2\mathbf{v} \end{aligned}$$

The energy equation is
Equation:

$$\rho\frac{DU}{Dt} = \nabla \bullet (k\nabla T) + Y,$$

and for a Newtonian fluid
Equation:

$$Y = -4\mu\Phi.$$

Because the velocity field is solenoidal, the velocity can be expressed as the curl of a vector potential.
Equation:

$$\mathbf{v} = \nabla \times \mathbf{A}.$$

The Laplacian of the vector potential can be expressed in terms of the vorticity.
Equation:

$$\begin{aligned} \mathbf{w} &= \nabla \times \mathbf{v} \\ &= \nabla \times (\nabla \times \mathbf{A}) \\ &= \nabla(\nabla \bullet \mathbf{A}) - \nabla^2\mathbf{A} \\ &= -\nabla^2\mathbf{A}, \text{ if } \nabla \bullet \mathbf{A} = 0 \end{aligned}$$

If the body force is conservative, i.e., gradient of a scalar, the body force and pressure can be eliminated from the Navier-Stokes equation by taking the curl of the equation.
Equation:

$$\frac{D\mathbf{w}}{Dt} = \mathbf{w} \bullet \nabla\mathbf{v} + v\nabla^2\mathbf{w}$$

where ν is the kinematic viscosity.
Isochoric motion is a restriction that has to be justified. Because it is justified in so many cases, it is easier to identify the cases when it does not apply. We showed during the discussion of the effects of compressibility that compressibility or non-isochoric is important in the cases of significant Mach number, high frequency oscillations such as in acoustics, large dimensions such as in meteorology, and motions with significant viscous or compressive heating.

- **(ib)** Irrotational motion. The velocity field is irrotational
  Equation:

$$\mathbf{w} = \nabla \times \mathbf{v} = 0$$

It follows that there exists a velocity potential (x,t) from which the velocity can be derived as
Equation:

$$\mathbf{v} = \nabla \phi$$

and in place of the three components of velocity we seek only one scalar function. (Note that some authors express the velocity as the gradient of a scalar and others as the negative of a gradient of a scalar. We will use either to conform to the book from which it was extracted.) The continuity equation becomes
Equation:

$$\frac{D\rho}{Dt} + \rho\, \nabla^2 \phi = 0$$

so that for an isochoric (or incompressible), irrotational motion, is a potential function satisfying
Equation:

$$\nabla^2 \phi = 0$$

The Navier-Stokes equations become
Equation:

$$\nabla \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 \right] = \mathbf{f} - \frac{1}{\rho} \nabla p + \left(\lambda' + 2\nu \right) \nabla \left(\nabla^2 \phi \right)$$

In the case of an irrotational body force $\mathbf{f} = -\nabla \Omega$ and when p is a function only of , this has an immediate first integral since every term is a gradient. Thus if $P\left(\rho\right) = \int dp/\rho$,
Equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \Omega + P\left(\rho\right) - \left(\lambda' + 2\nu \right) \nabla^2 \phi = g\left(t\right)$$

is a function of time only.
Irrotational motions with finite viscosity are only very special motions because the no-slip boundary conditions on solid surfaces usually will cause generation of rotation. Usually irrotational motion is associated with inviscid fluids because the no-slip boundary condition then will not apply and initially irrotational motion will remain irrotational.

- **(ic)** Complex lamellar motions, Betrami motions, ect. These names can be applied when the velocity field is of this type. Various simplifications are possible by expressing the velocity in terms of scalar fields. We shall not discuss them further here.
- **(id)** Plane flow. Here the motion is restricted to two dimensions which may be taken to be the 012 plane. Then v3 = 0 and x3 does not occur in the equations. Also, the vector potential and the vorticity have only one nonzero component.
  Incompressible plane flow. Since the flow is solenoidal, the velocity can be expressed as the curl of the vector potential. The nonzero component of the vector potential is the stream function.
  Equation:

$$\begin{aligned} \mathbf{v} &= \nabla \times \mathbf{A} \\ v_i &= \varepsilon_{ij3}\, A_{3,j} = \varepsilon_{ij}\, \psi_{,j} \\ (v_1,\ v_2,\ v_3) &= \left(\frac{\partial \psi}{\partial x_2},\ -\frac{\partial \psi}{\partial x_1},\ 0 \right) \end{aligned}$$

  The vorticity has only a single component, that in the 03 direction, which we will write without suffix
  Equation:

$$\begin{aligned} \mathbf{w} &= \nabla \times \mathbf{v} \\ w_i &= \varepsilon_{ijk}\, v_{k,j}, \quad j,k = 1,2 \\ w &= \tfrac{\partial v_2}{\partial x_1} - \tfrac{\partial v_1}{\partial x_2} \\ &= -\nabla^2 \psi \end{aligned}$$

  If the body force is conservative, i.e., gradient of a scalar, then the body force and pressure disappear from the Navier-Stokes equation upon taking the curl of the equations. In plane flow
  Equation:

$$\frac{Dw}{Dt} = \nu\, \nabla^2 w$$

  Thus for incompressible, plane flow with conservative body forces, the continuity equation and equations of motion reduce to two scalar equations.
  Incompressible, irrotational plane motion. A vector field that is irrotational can be expressed as the gradient of a scalar. Since the flow is incompressible, the velocity vector field is solenoidal and the Laplacian of the scalar is zero, i.e., it is harmonic or an analytical function.
  Equation:

$$\begin{aligned} \mathbf{v} &= \nabla\phi \\ \nabla \bullet \mathbf{v} &= 0 \\ &= \nabla^2\phi \end{aligned}$$

Since the flow is incompressible, it also can be expressed as the curl of the vector potential, or in plane flow as derivatives of the stream function as above. Since the flow is irrotational, the vorticity is zero and the stream function is also an analytical function, i.e., $\mathbf{v} = \nabla \times \mathbf{A}, \quad 0 = \mathbf{w} = \nabla \times \mathbf{v} = -\nabla^2\mathbf{A} + \nabla(\nabla \bullet \mathbf{A}), \Rightarrow \nabla^2\psi = 0$. Thus
Equation:

$$\begin{aligned} v_1 &= \frac{\partial\phi}{\partial x_1} &= \frac{\partial\psi}{\partial x_2} \\ v_2 &= \frac{\partial\phi}{\partial x_2} &= -\frac{\partial\psi}{\partial x_1} \end{aligned}$$

These relations are the Cauchy-Riemann relations show that the complex function $f = \phi + i\,\psi$ is an analytical function of $z = x_1 + i\,x_2$. The whole resources of the theory of functions of a complex variable are thus available and many solutions are known.
Steady, plane flow. If the fluid is compressible but the flow is steady (i.e., no quantity depends on t) the equation of continuity becomes
Equation:

$$\frac{\partial(\rho\,v_1)}{\partial x_1} + \frac{\partial(\rho\,v_2)}{\partial x_2} = 0$$

A stream function can again be introduced, this time in the form
Equation:

$$v_1 = \frac{1}{\rho}\frac{\partial\psi}{\partial x_2}, v_2 = -\frac{1}{\rho}\frac{\partial\psi}{\partial x_1}$$

The vorticity is now given by
Equation:

$$\rho\,w = -\nabla^2\psi + \frac{\nabla\psi \bullet \nabla\rho}{\rho}$$

- **(ie)** Axisymmetric flows. Here the flow has an axis of symmetry such that the flow field can be expressed as a function of only two coordinates by using curvilinear coordinates. The curl of the vector potential and velocity has only one non-zero component and a stream function can be found.
- **(if)** Parallel-flow perpendicular to velocity gradient. If the flow is parallel, i.e., the streamlines are parallel and are perpendicular to the velocity gradient, then the equations of motion become linear in velocity if the fluid is Newtonian.
  Equation:

$$\text{If} \quad \mathbf{v} \bullet \nabla \mathbf{v} = 0, \quad \text{then} \quad \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t}, \quad \text{and}$$
$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{f} + \nabla \bullet \mathbf{T}$$

The second type of specializations are limiting cases of the equations of motion.

- **(iia)** Hydrostatics. When there is no flow, the only non-zero terms in the equations of motion are the body forces and pressure gradient.
  Equation:

$$\rho \, \mathbf{f} = \nabla p$$

  If the body force is conservative, i.e., the gradient of a scalar, then the hydrostatic pressure can be determined from this scalar.
  Equation:

$$\begin{aligned} &\mathbf{f} = \nabla \Omega \\ &\nabla p = \rho \, \nabla \Omega \\ &\nabla(\Phi - \Omega) = 0 \\ &\Phi = \Omega + \text{constant} \\ &\text{where} \\ &\Phi = \int^{p} \frac{dp}{\rho} \end{aligned}$$

  If the body force is due to gravity then
  Equation:

$$\Omega = g \, z$$

  where z is the elevation above a datum such as the mean sea level.
- **(iib)** Steady flow. Examples of this have already been given and indeed it might have been considered as a restriction of the first class. All partial derivatives with respect to time vanish and the material derivative reduce to the following
  Equation:

$$\frac{D}{Dt} = \mathbf{v} \bullet \nabla$$

  In particular, the continuity equation is
  Equation:

$$\nabla \bullet (\rho \, \mathbf{v}) = 0$$

  so that the mass flux field is solenoidal.
- **(iic)** Creeping flow. It is sometimes justifiable to assume that the velocity is so small that the square of velocity is negligible by comparison with the velocity itself. This linearizes

the equations and allows them to be solved more readily. For example, the Navier-Stokes equation becomes
Equation:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{f} - \nabla p + (\lambda + \mu) \nabla (\nabla \bullet \mathbf{v}) + \mu \nabla^2 \mathbf{v}$$

In particular, for steady, incompressible, creeping flow with conservative body forces
Equation:

$$\begin{aligned} &\nabla P = \mu \nabla^2 \mathbf{v} \\ &\text{where} \\ &P = p - \rho\, g\, z \end{aligned}$$

However, since the continuity equation is $\nabla \bullet \mathbf{v} = 0$, we have
Equation:

$$\nabla^2 P = 0$$

or P is a harmonic potential function. This is the starting point for Stokes' solution of the creeping flow about a sphere and for its various improvements.
One may ask, "How small must the velocity be in order to neglect the nonlinear terms?" To answer this question, we need to examine the value of the Reynolds number. However, this time the pressure and body forces will be made dimensionless with respect to the shear forces rather than the kinetic energy.
Equation:

$$\begin{aligned} &\mathbf{v}^* = \tfrac{\mathbf{v}}{U}, \quad \mathbf{x}^* = \tfrac{\mathbf{x}}{L}, \quad t^* = \tfrac{U}{L} t, \quad P^{**} = \tfrac{P}{\mu U / L} \\ &\nabla^* = L\, \nabla, \quad \nabla^{*\,2} = L^2\, \nabla^2 \end{aligned}$$

The dimensionless Navier-Stokes equation for incompressible flow is now as follows
Equation:

$$\begin{aligned} N_{\mathrm{Re}} \tfrac{D\mathbf{v}^*}{Dt^*} &= -\nabla^* P^{**} + \nabla^{*\,2} \mathbf{v}^* \\ &\text{where} \\ N_{\mathrm{Re}} &= \tfrac{\rho\, U\, L}{\mu} = \tfrac{\rho\, U^2}{\mu\, U / L} \end{aligned}$$

Creeping flow is justified if the Reynolds number is small enough to neglect the left-hand side of the above equation. If the dimensionless variables and their derivatives are the order of unity, then creeping flow is justified if the Reynolds number is small compared to unity.
Another specialization of the equations of motion where the equations are made linear arises in stability theory when the basic flow is known but perturbed by a small amount. Here it is the squares and products of small perturbations that are regarded as negligible.

- **(iid)** <u>Inertial flow.</u> The flow is said to be inviscid when the inertial terms are dominant and the terms with viscosity in the equations of motion can be neglected. We can examine the conditions when this may be justified from the dimensionless equations of motion with the pressure and body forces normalized with respect to the kinetic energy.
  Equation:

$$\begin{aligned} N_{\mathrm{Re}}\left[\frac{D\mathbf{v}^*}{Dt^*}+\nabla^* P^*\right] &= (\lambda/\mu+1)\,\nabla^*\,\Theta^*+\nabla^{*2}\mathbf{v}^* \\ &\text{where} \\ N_{\mathrm{Re}} &= \frac{\rho U L}{\mu}=\frac{\rho U^2}{\mu U/L} \\ P^* &= \frac{P}{\rho U^2} \end{aligned}$$

  The limit of inviscid flow may occur when the Reynolds number is becomes very large such that the right-hand side of the above equation is negligible. Notice that if the right-hand side of the equation vanishes, the equation goes from being second order in spatial derivatives to first order. The differential equation goes from being parabolic to being hyperbolic and the number of possible boundary condition decreases. The no-slip boundary condition at solid surfaces can no longer apply for inviscid flow. These types of problems are known as singular perturbation problems where the differential equations are first order except near boundaries where they become second order. Physically, the form drag dominates the skin friction in inertial flow. The macroscopic momentum balance is described by Bernoulli theorems.
  Notice that inviscid flow is necessary for irrotational flow past solid objects but inviscid flow may be rotational. If fact much of the classical fluid dynamics of vorticity is based on inviscid flow.
- **(iie)** <u>Boundary-layer flows.</u> Ideal fluid or inviscid flow may be assumed far from an object but real fluids have no-slip boundary conditions on solid surfaces. The result is a *boundary-layer* of viscous flow with a large vorticity in a thin layer near solid surfaces that merges into the ideal fluid flow at some distance from the solid surface. It is possible to neglect certain terms of the equations of motion compared to others. The basic case of steady incompressible flow in two dimensions will be outlined. If a rigid barrier extends along the positive 01 axis the velocity components v_1 and v_2 are both zero there. In the region distant from the axis the flow is $v_1 = U\left(x_1\right)$, $v_2 \approx 0$, and may be expected to be of the form shown in Fig. 6.1, in which v_1 differs from U and v_2 from zero only within a comparatively short distance from the plate. To express this we suppose L is a typical dimension along the plate and δ a typical dimension of this boundary layer and that V_1 and V_2 are typical velocities of the order of magnitude of v_1 and v_2. We then introduce dimensionless variables
  Equation:

$$x^* = \frac{x_1}{L},\ y^* = \frac{x_2}{\delta},\ u^* = \frac{v_1}{V_1},\ v^* = \frac{v_2}{V_2}$$

  which will be of the order of unity. This in effect is a stretching upward of the coordinates so that we can compare orders of magnitude of the various terms in the equations of motion, for now all dimensionless quantities will be of order of unity. It is assumed that

$\delta << L$, but the circumstances under which this is valid will become apparent later. It is also assumed that the functions are reasonably smooth and no vast variations of gradient occur. The equation of continuity becomes
Equation:

$$\frac{V_1}{L}\frac{\partial u^*}{\partial x^*} + \frac{V_2}{\delta}\frac{\partial v^*}{\partial y^*} = 0$$

which would lose its meaning if one of these terms were completely negligible in comparison with the other. It follows that
Equation:

$$V_2 = O\left(V_1\,\frac{\delta}{L}\right)$$

where the symbol O means "is of the order of." The Navier-Stokes equations become
Equation:

$$\frac{{V_1}^2}{L}u^*\,\frac{\partial u^*}{\partial x^*} + \frac{V_1\,V_2}{\delta}\,v^*\,\frac{\partial u^*}{\partial y^*} = -\frac{p_o}{\rho\,L}\,\frac{\partial p^*}{\partial x^*} + \nu\frac{V_1}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{L^2}{\delta^2}\,\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

and

$$\frac{V_1\,V_2}{L}u^*\,\frac{\partial v^*}{\partial x^*} + \frac{V_2^2}{\delta}v^*\,\frac{\partial v^*}{\partial y^*} = -\frac{p_o}{\rho\,\delta}\,\frac{\partial p^*}{\partial y^*} + \nu\frac{V_2}{\delta^2}\left(\frac{\delta^2}{L^2}\,\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

In the first of these equations the last term on the right-hand side dominates the Laplacian and $\partial^2 u^*/\partial x^{*2}$ can be neglected. Dividing through by $V_1{}^2/L$ we see that the other terms will be of the same order of magnitude provided
Equation:

$$p_o = \rho\,V_1^2 \quad \text{and} \quad \frac{L\,\nu}{\delta^2\,V_1} = O\,(1)$$

Inserting these orders of magnitude in the second equation and the condition that $\delta << L$ results in the pressure gradient $\partial p/\partial y$ being the dominant term. Thus p is a function of x only. Returning to the original variables, we have the equations
Equation:

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0$$

$$v_1\,\frac{\partial v_1}{\partial x_1} + v_2\,\frac{\partial v_1}{\partial x_2} = -\frac{1}{\rho}\,\frac{dp}{dx} + \nu\frac{\partial^2 v_1}{\partial x_2^2}$$

The circumstances under which these simplified equations are valid are given by the term above that we assumed to be of order of unity, which we rewrite as
Equation:

$$\frac{\delta}{L} = O\left(\sqrt{\frac{\nu}{V_1\,L}}\right)$$

Since it was assumed that $\delta << L$ this equation shows that this will be the case if $v \ll V_1 L$. The assumption that the pressure is $O(\rho V_{1^2})$ is consistent with the assumption that the outer flow is inertial denominated flow.

- **(iif)** <u>Lubrication and film flow.</u> Lubrication and film flow is another case where one dimension is small compared to the other dimensions. Lubrication flow is usually between two solid surfaces with the relative velocity between the two surfaces specified. Film flow is usually between a solid and fluid surfaces or between two fluid interfaces and the driving force for flow is usually buoyancy or the Laplace pressure between curved interfaces. Because of the one dimension being small, lubrication and film flow is about always at low Reynolds numbers.
  The equations of motion are specialized by normalizing the variables with characteristic quantities of the flow and dropping the terms that are negligible. Incompressible flow with low Reynolds number of a Newtonian fluid is assumed. Let the direction normal to the film be the 03 direction. The double subscript, 12, will be used to denote the two coordinate directions in the plane of the film. Let h_o be a characteristic film thickness and L be a characteristic dimension in the plane of the film and $h_o/L << 1$. The variables are normalized to be order of unity.
  Equation:

$$\overset{*}{x}_{12} = \frac{x_{12}}{L}, \quad \overset{*}{x}_3 = \frac{x_3}{h_o}, \quad h^* = \frac{h}{h_o}, \quad t^* = \frac{t}{t_o}$$
$$\overset{*}{v}_{12} = \frac{v_{12}}{U}, \quad \overset{*}{v}_3 = \frac{v_3\, t_o}{h_o}, \quad P^* = \frac{p - \Delta\rho\, g\, z}{P_o}$$
$$\text{where}$$
$$P_o = \Delta\rho\, g\, L \quad \text{or} \quad P_o = \frac{\sigma}{R} \quad or \quad P_o = \mathrm{f}\left(\frac{\mu U}{L}\ \frac{L}{h_o}\right)$$

The film boundaries are material surfaces and the velocity perpendicular to the plane of the film is the rate of change of film thickness.
Equation:

$$v_3\big|_h = \frac{\partial h}{\partial t}$$
$$= \frac{h_o}{t_o}\frac{\partial h^*}{\partial t^*}$$

Substitution into the integral of the equation of continuity result in the following.
Equation:

$$\int_0^h \frac{\partial v_3}{\partial x_3}\, dx_3 = v_3\big|_0^h$$
$$\int_0^h \frac{\partial v_{12}}{\partial x_{12}}\, dx_3 = \frac{\partial (h\, v_{12})}{\partial x_{12}} + O\left(h_o/L\right)$$
$$\frac{\partial\left(h^*\, \overset{*}{v}_{12}\right)}{\partial \overset{*}{x}_{12}} + \left[\frac{L}{U\, t_o}\right]\frac{\partial h^*}{\partial t^*} = O\left(h_o/L\right)$$

The characteristic time can be chosen as to make the dimensionless group equal to unity.

Equation:

$$t_o = \frac{L}{U}$$

The dimensionless variables can now be substituted into the equations of motion for zero Reynolds number with gravitational body force and Newtonian fluid. For the components in the plane of the film,
Equation:

$$0 = -\nabla_{12}^{*} P^{*} + \left[\frac{\mu U}{P_o L}\right] \nabla_{12}^{*2} v_{12}^{*} + \left[\frac{\mu U L}{P_o h_o^2}\right] \frac{\partial^2 v_{12}^{*}}{\partial x_3^{*2}}$$

The dimensionless group in the last term can be made equal to unity because of the two characteristic quantities, P_o and U, only one is specified and the other can be determined from the first.
Equation:

$$\frac{\mu U L}{P_o h_o^2} = 1$$
$$\text{and}$$
$$U = \frac{P_o h_o^2}{L \mu}, \quad \text{or} \quad P_o = \frac{L \mu U}{h_o^2}$$

The equations of motion in the plane of the film is now
Equation:

$$0 = -\nabla_{12}^{*} P^{*} + \frac{\partial^2 v_{12}^{*}}{\partial x_3^{*2}} + O(h_o/L)^2$$

The dimensionless variables can now be substituted into the equation of motion perpendicular to the plane of the film.
Equation:

$$0 = -\frac{\partial P^{*}}{\partial x_3^{*}} + \left[\frac{\mu U L}{P_o h_o^2}\right] \left[\frac{h_o}{L}\right]^4 \nabla_{12}^{*2} v_3^{*} + \left[\frac{\mu U L}{P_o h_o^2}\right] \left[\frac{h_o}{L}\right]^2 \frac{\partial^2 v_3^{*}}{\partial x_3^{*2}}$$
$$0 = -\frac{\partial P^{*}}{\partial x_3^{*}} + O(h_o/L)^2$$

This last equation shows that the pressure is approximately uniform over the thickness of the film. Thus the velocity profile of v_{12}^{*} can be determined by integrating the equation of motion in the plane of the film twice and applying the appropriate boundary conditions. The average velocity in the film can be determined by integrating the velocity profile across the film.
Certain of the specializations based on the constitutive equation or equation of state have turned up already in the previous cases. We mention here a few important cases.

- **(iiia)** Incompressible fluid. An incompressible fluid is always isochoric and the considerations of (ia) apply. It should be remembered that for an incompressible fluid the pressure is not defined thermodynamically, but is an variable of the motion.
- **(iiib)** Perfect fluid. A perfect fluid has no viscosity so that
  Equation:

$$\mathbf{T} = -p\mathbf{I}$$
$$\text{and}$$
$$\rho\,\frac{D\mathbf{v}}{Dt} = \rho\,\mathbf{f} - \nabla p$$

  If, in addition, the fluid has zero conductivity the energy equation becomes
  Equation:

$$\frac{DS}{Dt} = 0$$

  and the flow is *isentropic*.
- **(iiic)** Ideal gas. An ideal gas is a fluid with the equation of state
  Equation:

$$p = \rho\,R\,T$$

  The entropy of an ideal gas is given by
  Equation:

$$S = \int c_v\,\frac{dT}{T} - R\ \ln\ \rho$$

  which for constant specific heats gives
  Equation:

$$p = e^{S/c_v}\,\rho^{\gamma}$$

- **(iiid)** Piezotropic fluid and barotropic flow. When the pressure and density are directly related, the fluid is said to be piezotropic. A simple relation between p and allows us to write
  Equation:

$$\frac{1}{\rho}\nabla p = \nabla P\left(\rho\right) = \nabla\int^{p}\frac{dp}{\rho}$$

- **(iiie)** Newtonian fluids. Here the assumption of a linear relation between stress and strain leads to the constitutive equation
  Equation:

$$\mathbf{T} = \left(-p + \lambda\,\Theta\right)\mathbf{I} + 2\mu\,\mathbf{e}$$

The equations of motion becomes the Navier-Stokes equations.

Assignment 6.3
For 2-D, incompressible flow, prove that $\psi = \int \mathbf{n} \bullet \mathbf{v}\, ds$

Assignment 6.4
Carry out the steps in specializing the continuity and equations of motion for boundary layer and lubrication or film flows.

Assignment 6.5
Derive the equations for the propagation of acoustic waves.

Assignment 6.6
Derive the simplifications that are possible to the continuity equation and equations of motion of Newtonian fluids for the following cases:

1. Incompressible flow
2. Irrotational flow
3. Incompressible and irrotational flow
4. Low Reynolds number.
5. One dimension small compared to other dimensions.
6. Laminar boundary layer.

Boundary conditions

The flow field is often desired for a finite region of space that is bounded by a surface. Boundary conditions are needed on these surfaces and at internal interfaces for the flow field to be determined. The boundary conditions for temperature and heat flux are continuity of both across internal interfaces that are not sources or sinks and either a specified temperature, heat flux, or a combination of both at external boundaries.

Surfaces of symmetry. Surfaces of symmetry corresponds to reflection boundary conditions where the normal component of the gradient of the dependent variables are zero. Thus surfaces of symmetry have zero momentum flux, zero heat flux, and zero mass flux. Because the momentum flux is zero, the shear stress is zero across a surface of symmetry.

Periodic boundary. Periodic boundaries are boundaries where the dependent variables and its derivatives repeat themselves on opposite boundaries. The boundaries may or may not be symmetry boundary conditions. An example of when periodic boundaries are not symmetry boundaries are the boundaries of $\theta = 0$ and $\theta = 2\pi$ of a non-symmetric system with cylindrical polar coordinate system.

Solid surfaces. A solid surface is a material surface and kinematics require that the mass flux across the surface to be zero. This requires the normal component of the fluid velocity to be that of the solid. The tangential component of velocity depends on the assumption made about the fluid viscosity. If the fluid is assumed to have zero viscosity the order of the equations of motion reduce to first order and the tangential components of velocity can not be specified.

Viscous fluids stick to solid surfaces and the tangential components of velocity is equal to that of the solid. Exception to the 'no-slip' boundary conditions is when the mean free path of as gas is similar to the dimensions of the solid. An example is the flow of gas through a fine pore porous media.

Porous surface. A porous surface may not be a no-flow boundary. Flux through a porous material is generally described by Darcy's law.

Fluid surfaces. If there is no mass transfer across a fluid-fluid interface, the interface is a material surface and the normal component of velocity on either side of the interface is equal to the normal component of the velocity of the interface. The tangential component of velocity at a fluid interface is not known apriori unless the interface is assumed to be immobile as a result of adsorbed materials. The boundary condition at fluid interfaces is usually jump conditions on the normal and tangential components of the stress tensor. Aris give a thorough discussion on the dynamical connection between the surface and its surroundings. If we assume that the interfacial tension is constant and that it is possible to neglect the surface density and the coefficients of dilational and shear surface viscosity then the jump condition across a fluid-fluid interface is

Equation:

$$[T_{ij}]\, n_j = -2H\,\sigma\, n_i$$
$$[\mathbf{T}] \bullet \mathbf{n} = -2H\,\sigma\,\mathbf{n}$$

where the bracket denotes the jump condition across the interface, H is the mean curvature of the interface, and σ is the interfacial or surface tension. The jump condition on the normal component of the stress is the jump in pressure across a curved interface given by the Laplace-Young equation. The tangential components of stress are continuous if there are no surface tension gradients and surface viscosity. Thus the tangential stress at the clean interface with an inviscid fluid is zero.

For boundary conditions at a fluid interface with adsorbed materials and thus having interfacial tension gradients and surface viscosity, see Chapter 10 of Aris and the thesis of Singh (1996).

Boundary conditions for the potentials and vorticity. Some fluid flow problems are more conveniently calculated through the scalar and vector potentials and the vorticity.

Equation:

$$\mathbf{v} = -\nabla\phi + \nabla \times \mathbf{A}$$
$$\nabla^2\phi = -\nabla \bullet \mathbf{v}$$
$$\nabla^2\mathbf{A} = -\mathbf{w}$$

The boundary condition on the scalar potential is that the normal derivative is equal to the normal component of velocity.

Equation:

$$\begin{aligned}\mathbf{n} \bullet \nabla\phi &\equiv \frac{\partial\phi}{\partial n} \\ &= -\mathbf{n} \bullet \mathbf{v}\end{aligned}$$

The boundary condition on the vector potential is that the tangential components vanish and the normal derivative of the normal component vanish (Hirasaki and Hellums, 1970). Wong and Reizes (1984) introduced a method where the need for calculation of the scalar potential is replaced by the use of an irrotational component of velocity.
Equation:

$$\begin{aligned}\mathbf{A}_{(t)} &= 0 \\ \frac{\partial \mathbf{A}_{(n)}}{\partial n} &= 0\end{aligned}$$

In two dimensional or axisymmetric incompressible flow, it is not necessary to have a scalar potential and the single nonzero component of the vector potential is the stream function. The boundary condition on the stream function for flow in the x_1, x_2 plane of Cartesian coordinates is
Equation:

$$\begin{aligned}\mathbf{n} \bullet \mathbf{v} &= \mathbf{n} \bullet \nabla \times \mathbf{A} \\ &= n_1 \frac{\partial\psi}{\partial x_2} - n_2 \frac{\partial\psi}{\partial x_1} \\ &\text{where} \\ \psi &= A_3\end{aligned}$$

The boundary condition on the normal component of the vorticity of a fluid with a finite viscosity on a solid surface is determined from the tangential components of the velocity of the solid. It is zero if the solid is not rotating. If the boundary is an interface between two viscous fluids then the normal component of vorticity is continuous across the interface (C. Truesdell, 1960). If the interface is with an inviscid fluid, the tangential components of vorticity vanishes and the normal derivative of the normal component vanishes for a plane interface (Hirasaki, 1967).

If the boundary is at along a region of space for which the velocity field is known, the vorticity can be calculated from the derivatives of the velocity field.

If the boundary is a surface of symmetry, the tangential components of vorticity must vanish because the normal component of velocity and the normal derivative of velocity vanish. The normal derivative of the normal component of vorticity vanishes from the solenoidal property of vorticity.

Scaling, Dimensional Analysis, and Similarity

We have already seen some examples of scaling and dimensional analysis when we determined when the continuity equations and equations of motion could be simplified. The concept of similarity states that the solution of transport problems do not need to be determined separately for each value of the parameters. Rather the variables and parameters can be grouped into dimensionless variables and dimensionless numbers and the solution will have fewer degrees of freedom. Also, in some cases the partial differential equations can have the independent variables combined to fewer independent variables and be expressed as ordinary differential equations. The concept of similarity does not apply only to mathematical solutions but is also used to design physical analogs of systems on a smaller scale or with different transport mechanism. For example, before numerical simulation the streamlines and pressure gradients for flow in petroleum reservoirs were studied by electrical conduction on a laboratory scale model that is geometrically similar.

Dimensionless groups based on geometry. The aspect ratio is the ratio of the characteristic lengths of the system. The symbol α is normally used to denote the aspect ratio. We saw how the aspect ratio simplified the equations of motion for boundary layer flow and lubrication or film flow. The table below lists some examples of aspect ratio expressions for different problems.

Examples of aspect ratio, α	
δ/L	boundary layer flow
h_o/L	lubrication or film flow
L_y/L_x	width/length
D/L	entrance effects in pipe flow
D_{max}/D_{min}	eccentricity ratio

It may be difficult to define the characteristic length of an irregularly shaped conduit or object. The characteristic dimension for an irregular conduit or object can be determined by the *hydraulic diameter*.
Equation:

$$D_h = \begin{cases} \frac{4A_x}{P_w}, & \text{conduit} \\ \frac{4V}{A_w}, & 2-\text{D object} \\ \frac{6V}{A_w}, & 3-\text{D object} \end{cases}$$

where A_x and P_w are the cross-sectional area and wetted perimeter of the conduit and V and A_w are the volume and wetted area of an object. The definitions reduce to the diameter of a cylinder or sphere for regular objects. (The reader should be aware that some definitions such as the hydraulic radius in BSL may not reduce to the dimension of the regular object.) The hydraulic diameter may provide length scales but exact similarity is not satisfied unless the conduits or objects are geometrically similar.

Dimensionless groups based on equations of motion and energy. We derived earlier the Reynolds number from the equations of motion and dimensionless groups from the energy equation when compressibility is important. We will discuss these further and additional dimensionless groups.

Interpretation of the Reynolds number	
$\frac{\rho U L}{\mu}$	basic definition
$\frac{\rho U^2}{\mu U/L}$	$\frac{\text{kinetic energy}}{\text{shear stress}}$
$\approx \frac{\lvert\rho \mathbf{v}\bullet(\nabla\mathbf{v})\rvert}{\lvert\mu\nabla^2\mathbf{v}\rvert}$	$\frac{\text{inertial force}}{\text{viscous force}}$

The Reynolds number can be interpreted as a ratio of kinetic energy to shear stress. We will see later that some of the dimensionless numbers differ by whether it is normalized with respect to the kinetic energy or the shear stress term, i.e., it is a product of the Reynolds number and another dimensionless number.

Suppose the characteristic value of the body force is $\rho g L$ and the characteristic value of pressure is σ/L , i.e., due to capillary forces. The dimensionless Navier-Stokes equation can be expressed as follows.

Equation:

$$\frac{Dv^*}{Dt^*} = \left[\frac{g\,L}{U^2}\right]\left(\frac{\mathbf{f}}{g}\right) - \left[\frac{\sigma}{\rho\,U^2\,L}\right]\nabla^* p^* + \left[\frac{1}{N_{\mathrm{Re}}}\right]\nabla^{*2}\mathbf{v}^*$$

We can now define dimensionless groups that include gravity or buoyancy forces and gravity forces.

Dimensionless groups based on gravity and capillarity	
$N_{Fr} = \frac{U^2}{gL} = \frac{\rho U^2}{\rho g L}$	Froude number
$N_{We} = \frac{\rho U^2 L}{\sigma} = \frac{\rho U^2}{\sigma / L}$	Weber Number
$N_{Ca} = \frac{\mu U}{\sigma} = \frac{\mu U / L}{\sigma / L} = \frac{N_{We}}{N_{\mathrm{Re}}}$	capillary number
$\frac{\rho g L^2}{\mu U} = \frac{\rho g L}{\mu U / L} = \frac{N_{\mathrm{Re}}}{N_{Fr}}$	gravity number
$N_{Bo} = \frac{\rho g L^2}{\sigma} = \frac{\rho g L}{\sigma / L} = \frac{N_{We}}{N_{Fr}}$	Bond number

We derived scale factors that have to be small in order to neglect the effects of compressibility. They are summarized here.

Dimensionless groups necessary for incompressible flow	
$N_{Ma} = \frac{U}{c}$	Mach number
$\frac{\nu L}{c}$	(frequency length)/sonic velocity
$\frac{gL}{c^2} = \frac{\rho g L}{\gamma p}$	density change due to body forces

Friction factor and drag coefficients. Friction factor and drag coefficients are the force on the wall of a conduit or on an object normalized with respect to kinetic energy. There are some ambiguities in the literature that one should be aware of.

Friction factors	
$f_{SP} = \frac{\tau_w}{\rho U_m^2}$	Stanton-Pannel
	Fanning

$f_F = \frac{2\tau_w}{\rho U_m^2}$	
$f_{DW} = \frac{8\tau_w}{\rho U_m^2}$	Darcy-Weisbach
$f_{Moody=} \frac{2D\,\Delta P/L}{\rho U_m^2} = 4f_F = f_{DW}$	Moody
$f_{drag} = \frac{F_{friction}+F_{drag}}{A_p(1/2\,\rho U_\infty^2)}$	drag coefficient
$C_t = \frac{F_{friction}+F_{drag}}{A_p\,\rho U_\infty^2}$	drag coefficient

Nomenclature of S. W. Churchill

A_p projected area, m^2

$C_f = \frac{F_f}{A_p\,\rho\,u_\infty^2}$, mean drag coefficient due to friction

$C_p = \frac{F_p}{A_p\,\rho\,u_\infty^2}$, mean drag coefficient due to pressure

$C_t = \frac{F_f+F_p}{A_p\,\rho\,u_\infty^2}$, total mean drag coefficient

F_f drag force due to friction, N

F_p drag force due to pressure, N

Bernoulli Theorems

When the viscous effects are negligible compared with the inertial forces (i.e., large Reynolds number) there are a number of generalizations that can be made about the flow. These are described by the Bernoulli theorems. The fluid is assumed to be inviscid and have zero thermal conduction so that the flow is also barotropic (density a single-valued function of pressure). The first of the Bernoulli theorems is derived for flow that may be rotational. A special case is for motions relative to a rotating coordinate system where Coriolis forces arise. For irrotational flow, the Bernoulli theorem is a statement of the conservation kinetic energy, potential energy, and the expansion energy. A macroscopic energy balance can be made that includes the effects of viscous dissipation and the work done by the system.

Steady, barotropic flow of an inviscid, nonconducting fluid with conservative body forces. The equations of motion for a Newtonian fluid is
Equation:

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\,\mathbf{f} - \nabla p + (\lambda+\mu)\nabla(\nabla\bullet\mathbf{v}) + \mu\nabla^2\mathbf{v}$$

The assumptions of steady, inviscid flow simplify the equations to
Equation:

$$\rho(\mathbf{v} \bullet \nabla)\mathbf{v} = \rho\,\mathbf{f} - \nabla p$$

The assumptions of barotropic flow with conservative body forces allow,
Equation:

$$\mathbf{f} = -\nabla\Omega, \qquad \Phi\,(p) = \int^{p} \frac{dp}{\rho}$$
$$(\mathbf{v} \bullet \nabla)\mathbf{v} = -\nabla(\Omega + \Phi(p))$$

and by virtue of the identity
Equation:

$$(\mathbf{v} \bullet \nabla)\mathbf{v} = \nabla\left(\tfrac{1}{2}v^2\right) + \mathbf{w} \times \mathbf{v}$$

the equations of motion can be written
Equation:

$$\begin{aligned}
&\nabla\left(\Omega + \Phi\,(p) + \tfrac{1}{2}v^2\right) = \mathbf{v} \times \mathbf{w} \\
&\nabla H = \mathbf{v} \times \mathbf{w} \\
&H \equiv \Omega + \Phi\,(p) + \tfrac{1}{2}v^2
\end{aligned}$$

If the body force is gravitational then $\Omega = \rho g z$.

Let H denote the function of which the gradient occurs on the left-hand side of this equation. ∇H is a vector normal to the surfaces of constant H. However, $\mathbf{v} \times \mathbf{w}$ is a vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ so that these vectors are tangent to the surface. However, $\mathbf{v}$ and $\mathbf{w}$ are tangent to the streamlines and vortex lines respectively so that these must lie in a surface of constant H. It follows that H is constant along the streamlines and vortex lines. The surfaces of constant H which are crossed with this network of stream and vortex lines are known as Lamb surfaces and are illustrated in Fig. 6.2.

<u>Coriolis force.</u> Suppose that the motion is steady relative to a steadily rotating axis with an angular velocity, ω . Batchelor (1967) derives the equations of motion in this rotating frame and shows that we must now include the potential from the centrifugal force and the addition of a Coriolis force term.
Equation:

$$\begin{aligned}
&\nabla H = \mathbf{v} \times (\mathbf{w} + 2\omega) \\
&H \equiv \Omega + \Phi\,(p) + \tfrac{1}{2}v^2 - \tfrac{(\omega \times \mathbf{x})^2}{2}
\end{aligned}$$

<u>Irrotational flow.</u> If the flow is also irrotational, then $\mathbf{w} = 0$ and hence the energy function

Equation:

$$H \equiv \Omega + \Phi\left(p\right) + \tfrac{1}{2}v^2$$

is constant everywhere.

Ideal gas. For an ideal gas we have
Equation:

$$p = \left(c_p - c_v\right)\rho T$$

and if the heat capacities are assumed constant an isentropic change of state results in the following expression for H.
Equation:

$$H_{\text{ideal gas}} = \Omega + c_p T + \tfrac{1}{2}v^2$$

The gas is hotter at places on a streamline where the speed is smaller or has a lower potential energy. This may represent the heating of air at a stagnation point, cooling of ascending air, or heating of descending air.

The transformation from the function of pressure $\Phi(p)$ to heat capacity and temperature in the above equation for the isoentropic expansion of an ideal gas with constant heat capacity is proven as follows. The total differential of entropy in terms of pressure and temperature is as follows.
Equation:

$$dS = \left(\frac{\partial S}{\partial p}\right)_T dp + \left(\frac{\partial S}{\partial T}\right)_p dT$$

In an isoentropic expansion, the change of entropy is zero and this gives us a relation between the differential of pressure and differential of temperature.
Equation:

$$\left(\frac{\partial S}{\partial p}\right)_T dp = -\left(\frac{\partial S}{\partial T}\right)_p dT, \quad \text{for } dS = 0$$

The coefficient on the left-hand side can be determined from the Maxwell relations and the ideal gas EOS.
Equation:

$$\begin{aligned}\left(\frac{\partial S}{\partial p}\right)_T &= -\left(\frac{\partial V}{\partial T}\right)_p, \quad \text{Maxwell relation} \\ &= -\frac{(c_p - c_v)}{p} = -\frac{1}{\rho T}, \quad \text{ideal gas}\end{aligned}$$

The coefficient of the right hand side can be determined from the definition of the heat capacity at constant pressure.
Equation:

$$\begin{aligned}c_p &= \left(\frac{\partial' q}{\partial T}\right)_p, \quad \text{reversible process} \\ &= T\left(\frac{\partial S}{\partial T}\right)_p \\ &\therefore \left(\frac{\partial S}{\partial T}\right)_p = \frac{c_p}{T}\end{aligned}$$

Substituting these relations into the equation gives us the following.
Equation:

$$\begin{aligned}\frac{dp}{\rho} &= c_p\, dT \\ \int^p \frac{dp}{\rho} &= \int^T c_p\, dT \\ &= c_p T, \quad \text{if } c_p \text{ is constant} \\ & Q.E.D.\end{aligned}$$

Solution of the Partial Differential Equations
Topics Covered in the Chapter

- Classes of partial differential equations
- Systems described by the Poisson and Laplace equation
- Systems described by the diffusion equation
- Greens function, convolution, and superposition
- Green's function for the diffusion equation
- Similarity transformation
- Complex potential for irrotational flow
- Solution of hyperbolic systems

Classes of partial differential equations

The partial differential equations that arise in transport phenomena are usually the first order conservation equations or second order PDEs that are classified as elliptic, parabolic, and hyperbolic. A system of first order conservation equations is sometimes combined as a second order hyperbolic PDE. The student is encouraged to read R. Courant, Methods of Mathematical Physics, Volume II Partial Differential Equations, 1962 for a complete discussion.

System of conservation laws. Denote the set of dependent variables (e.g., velocity, density, pressure, entropy, phase saturation, concentration) with the variable u and the set of independent variables as t and x, where x denotes the spatial coordinates. In the absence of body forces, viscosity, thermal conduction, diffusion, and dispersion, the conservation laws (accumulation plus divergence of the flux and gradient of a scalar) are of the form
Equation:

$$\frac{\partial g(u)}{\partial t} + \nabla \cdot f(u) = \frac{\partial g(u)}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial u}{\partial t} + \frac{\partial f}{\partial g}\frac{\partial u}{\partial x}$$

where

$$\frac{\partial f}{\partial g} = \left[\frac{\partial(g_1, g_2, \cdots g_n)}{\partial(u_1, u_2, \cdots, u_n)}\right]^{-1} \frac{\partial(f_1, f_2, \cdots, f_n)}{\partial(u_1, u_2, \cdots, u_n)}, \quad \text{Jacobian matrix}$$

This is a system of first order quasilinear hyperbolic PDEs. They can be solved by the method of characteristics. These equations arise when transport of material or energy occurs as a result of convection without diffusion.

The derivation of the equations of motion and energy using convective coordinates (Reynolds transport theorem) resulted in equations that did not have the accumulation and convective terms in the form of the conservation laws. However, by derivation of

the equations with fixed coordinates (as in Bird, Stewart, and Lightfoot) or by application of the continuity equation, the momentum and energy equations can be transformed so that the accumulation and convective terms are of the form of conservation laws. Viscosity and thermal conductivity introduce second derivative terms that make the system non-conservative. This transformation is illustrated by the following relations.

Equation:

$$
\begin{aligned}
\frac{D\rho}{Dt} + \rho\,\nabla \bullet \mathbf{v} &= \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\,\mathbf{v}) = 0, \quad \text{continuity equation} \\
\rho\frac{DF}{Dt} &= \rho\frac{\partial F}{\partial t} + \rho\,\mathbf{v} \bullet \nabla F, \quad \text{for } F = \mathbf{v},\ S,\ \text{or } E \\
\frac{\partial(\rho\,F)}{\partial t} &= \rho\frac{\partial F}{\partial t} + F\frac{\partial \rho}{\partial t} \\
\nabla \bullet (\rho\,F\,\mathbf{v}) &= \rho\,F\,\nabla \bullet \mathbf{v} + \mathbf{v} \bullet \nabla(\rho\,F) = \rho\,F\,\nabla \bullet \mathbf{v} + F\,\mathbf{v} \bullet \nabla\rho + \rho\,\mathbf{v} \bullet \nabla F \\
\rho\frac{DF}{Dt} &= \frac{\partial(\rho\,F)}{\partial t} - F\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\,F\,\mathbf{v}) - \rho\,F\,\nabla \bullet \mathbf{v} - F\,\mathbf{v} \bullet \nabla\rho \\
&= \frac{\partial(\rho\,F)}{\partial t} + \nabla \bullet (\rho\,F\,\mathbf{v}) - F\left[\frac{\partial \rho}{\partial t} + \rho\,\nabla \bullet \mathbf{v} + \mathbf{v} \bullet \nabla\rho\right] \\
&\therefore \\
\rho\frac{DF}{Dt} &= \frac{\partial(\rho\,F)}{\partial t} + \nabla \bullet (\rho\,F\,\mathbf{v}), \quad \text{Q.E.D.}
\end{aligned}
$$

Assignment 7.1

a. For the case of inviscid, nonconducting fluid, in the absence of body forces, derive the steps to express the continuity equation, equations of motion, and energy equations as conservation law equations for mass, momentum, the sum of kinetic plus internal energy, and entropy.
b. For the case of isentropic, compressible flow, express continuity equation and equations of motions in terms of pressure and velocity. Transform it to a second order hyperbolic equation in the case of small perturbations.

<u>Second order PDE.</u> The classification of second order PDEs as elliptic, parabolic, and hyperbolic arise from a transformation of the independent variables. The classification apply to quasilinear (i.e., linear in the highest order derivatives) but we will only discuss linear equations with constant coefficients here. Numerical solutions are needed for quasilinear systems. Again let u denote the dependent variables and t, x, y, z as the independent variables. Examples of the different classes of equations are

Equation:

$$
\begin{aligned}
0 &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \rho = \nabla^2 u + \rho, && \text{elliptic equation} \\
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \rho = \nabla^2 u + \rho, && \text{parabolic equation} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \rho = \nabla^2 u + \rho, && \text{hyperbolic equatio}\, n
\end{aligned}
$$

The ρ term represents sources. When the cgs system of units is used in electrostatics and ρ is the charge density, the source is expressed as $4\pi\,\rho$. The factor 4π is absent with the mks or SI system of units. The parabolic PDEs are sometimes called the diffusion equation or heat equation. In the limit of steady-state conditions, the parabolic equations reduce to elliptic equations. The hyperbolic PDEs are sometimes called the wave equation. A pair of first order conservation equations can be transformed into a second order hyperbolic equation.

<u>Convective-diffusion equation.</u> The above equations represented convection without diffusion or diffusion without convection. When both the first and second spatial derivatives are present, the equation is called the convection-diffusion equation.
Equation:

$$
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{1}{N_{Pe}} \frac{\partial^2 u}{\partial x^2}
$$

Usually a dimensionless group such as the Reynolds number, or Peclet number appears as a factor to quantify the relative contribution of convection and diffusion.

Systems described by the Poisson and Laplace equation

We saw earlier that an irrotational vector field can be expressed as the gradient of a scalar and if in addition the vector field is solenoidal, then the scalar potential is the solution of the Laplace equation.
Equation:

$$
\begin{aligned}
&\mathbf{v} = -\nabla\phi, && \text{irrotational flow} \\
&\nabla \bullet \mathbf{v} = \Theta = -\nabla^2\phi \\
&\nabla \bullet \mathbf{v} = 0 = -\nabla^2\phi, && \text{incompressible, irrotational flow}
\end{aligned}
$$

Also, if the velocity field is solenoidal then the velocity can be expressed as the curl of the vector potential and the Laplacian of the vector potential is equal to the negative of the vorticity. If the flow is irrotational, then the vorticity is zero and the vector potential is a solution of the Laplace equation.

Equation:

$$\begin{array}{ll}
\mathbf{v} = \nabla \times \mathbf{A}, & \text{incompressible flow} \\
\nabla \times \mathbf{v} = \mathbf{w} = -\nabla^2 \mathbf{A}, & \text{for } \nabla \bullet \mathbf{A} = 0 \\
\nabla^2 \psi = -w, & \text{in two dimensions} \\
\nabla \times \mathbf{v} = 0 = -\nabla^2 \mathbf{A}, & \text{irrotational flow and} \quad \nabla \bullet \mathbf{A} = 0 \\
\nabla^2 \psi = 0, & \text{for two dimensional, irrotational, incompressible flow}
\end{array}$$

Other systems, which are solution of the Laplace equation, are steady state heat conduction in a homogenous medium without sources and in electrostatics and static magnetic fields. Also, the flow of a single-phase, incompressible fluid in a homogenous porous media has a pressure field that is a solution of the Laplace equation.

Systems described by the diffusion equation

Diffusion phenomena occur with viscous flow, thermal conduction, and molecular diffusion. Heat conduction and diffusion without convection are described by the diffusion equation. Convection is always present in fluid flow but its contribution to the momentum balance is neglected for creeping (low Reynolds number) flow or cases where the velocity is perpendicular to the velocity gradient. In this case
Equation:

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{f} - \frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v}, \quad \text{velocity perpendicular to velocity gradient}$$

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \nabla^2 \mathbf{v}, \quad \text{if } \mathbf{f} \text{ and } \nabla p \text{ vanish}$$

Green's function, convolution, and superposition

A property of linear PDEs is that if two functions are each a solution to a PDE, then the sum of the two functions is also a solution of the PDE. This property of superposition can be used to derive solutions for general boundary, initial conditions, or distribution of sources by the process of convolution with a Green's function. The student is encouraged to read P. M. Morse and H. Feshbach, Methods of Theoretical Physics, 1953 for a discussion of Green's functions.

The Green's function is used to find the solution of an inhomogeneous differential equation and/or boundary conditions from the solution of the differential equation that is homogeneous everywhere except at one point in the space of the independent

variables. (The initial condition is considered as a subset of boundary conditions here.) When the point is on the boundary, the Green's function may be used to satisfy inhomogeneous boundary conditions; when it is out in space, it may be used to satisfy the inhomogeneous PDE.

The concept of Green's solution is most easily illustrated for the solution to the Poisson equation for a distributed source $\rho(x, y, z)$ throughout the volume. The Green's function is a solution to the homogeneous equation or the Laplace equation except at (x_o, y_o, z_o) where it is equal to the Dirac delta function. The Dirac delta function is zero everywhere except in the neighborhood of zero. It has the following property.
Equation:

$$\int_{-\infty}^{\infty} f(\xi)\, \delta(\xi - x)\, d\xi = f\,(x)$$

The Green's function for the Poisson equation in three dimensions is the solution of the following differential equation
Equation:

$$\begin{aligned} &\nabla^2 G = -\delta\,(\mathbf{x} - \mathbf{x}_o) = -\,\delta\,(x - x_o)\,\delta\,(y - y_o)\,\delta\,(z - z_o) \\ &G\,(\mathbf{x}|\mathbf{x}_o) = \tfrac{1}{4\pi|\mathbf{x}-\mathbf{x}_o|} \end{aligned}$$

It is a solution of the Laplace equation except at $\mathbf{x} = \mathbf{x}_o$ where it has a singularity, i.e., it has a point source. The solution of the Poisson equation is determined by convolution.
Equation:

$$u\,(\mathbf{x}) = \iiint G\,(\mathbf{x}|\mathbf{x}_o)\,\rho\,(\mathbf{x}_o)\,dx_o\,dy_o\,dz_o$$

Suppose now that one has an elliptic problem in only two dimensions. One can either solve for the Green's function in two dimensions or just recognize that the Dirac delta function in two dimensions is just the convolution of the three-dimensional Dirac delta function with unity.
Equation:

$$\delta\,(x - x_o)\,\delta\,(y - y_o) = \int_{-\infty}^{\infty} \delta\,(x - x_o)\,\delta\,(y - y_o)\,\delta\,(z - z_o)\,dz_o$$

Thus the two-dimensional Green's function can be found by convolution of the three dimensional Green's function with unity.
Equation:

$$\begin{aligned} G(x,y|x_o,y_o) &= \int_{-\infty}^{\infty} G(\mathbf{x}|\mathbf{x}_o)\,dz_o \\ &= -\tfrac{1}{4\pi}\ln\left[(x-x_o)^2+(y-y_o)^2\right] \end{aligned}$$

This is a solution of the Laplace equation everywhere except at (x_o, y_o) where there is a line source of unit strength. The solution of the Poisson equation in two dimensions can be determined by convolution.
Equation:

$$u(x,y) = \iint G(x,y|x_o,y_o)\,\rho(x_o,y_o)\,dx_o\,dy_o$$

Assignment 7.2 Derivation of the Green's function
Derive the Green's function for the Poisson equation in 1-D, 2-D, and 3-D by transforming the coordinate system to cylindrical polar or spherical polar coordinate system for the 2-D and 3-D cases, respectively. Compare the results derived by convolution.

Method of Images

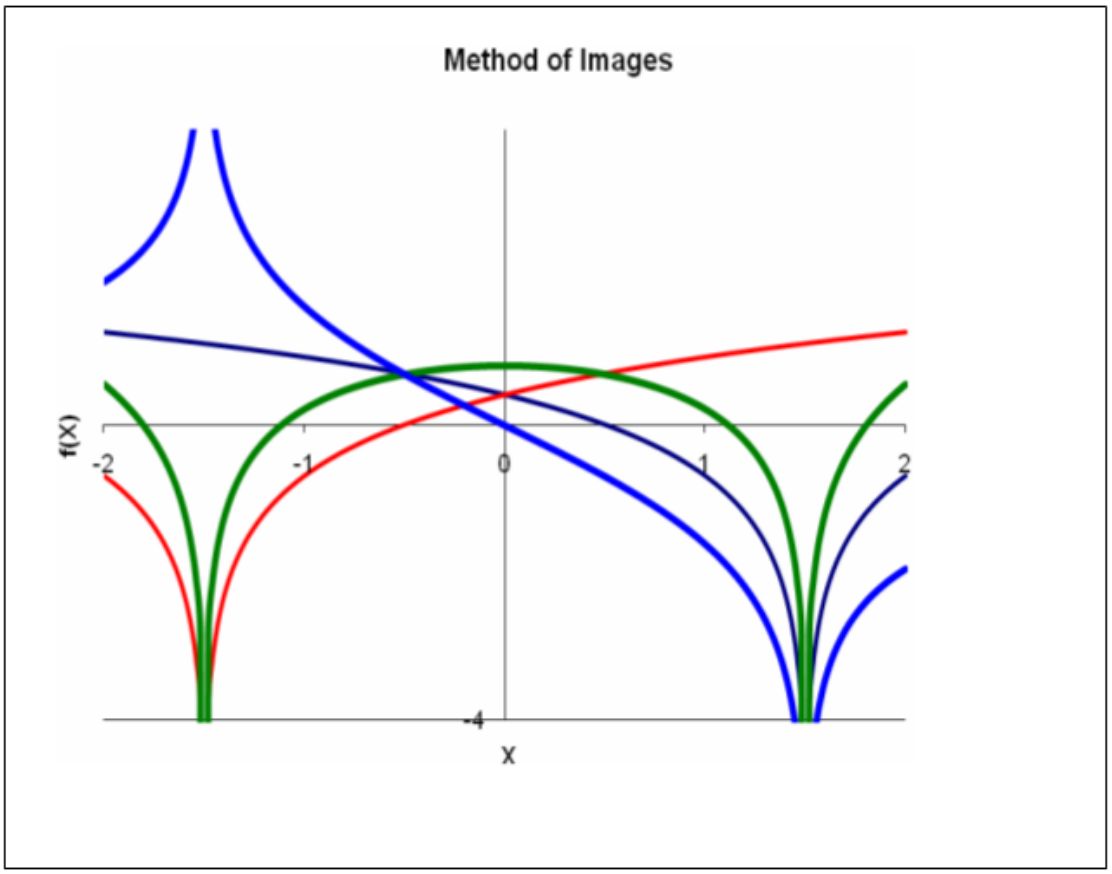
Green's functions can also be determined for inhomogeneous boundary conditions (the boundary element method) but will not be discussed here. The Green's functions discussed above have an infinite domain. Homogeneous boundary conditions of the Dirichlet type $(u=0)$ or Neumann type $(\partial u/\partial n = 0)$ along a plane(s) can be determined by the method of images.

Suppose one wished to find the solution to the Poisson equation in the semi-infinite domain, $y>0$ with the specification of either $u=0$ or $\partial u/\partial n = 0$ on the boundary, $y=0$. Denote as $u^0\,(x,y,z)$ the solution to the Poisson equation for a distribution of sources in the semi-infinite domain $y>0$. The solutions for the Dirichlet or Neumann boundary conditions at $y=0$ are as follows.
Equation:

$$\begin{aligned} u\,(x,y,z) &= u^0\,(x,y,z) - u^0\,(x,-y,z), \quad \text{for } u=0 \; at \; y=0 \\ u\,(x,y,z) &= u^0\,(x,y,z) + u^0\,(x,-y,z), \quad \text{for } du/dy=0 \; at \; y=0 \end{aligned}$$

The first function is an odd function of y and it vanishes at $y = 0$. The second is an even function of y and its normal derivative vanishes at $y = 0$.

An example of the method of images to satisfy either the Dirichlet or Neumann boundary conditions is illustrated in the following figure. The black curve is the response to a line sink at $x = 1.5$. We desire to have either the function or the derivative at $x = 0$ to vanish. The red curve is a line sink at $x = -1.5$. The sum of the two functions is symmetric about $x = 0$ and has zero derivative there. The difference is anti-symmetric about $x = 0$ and vanishes at $x = 0$.



Now suppose there is a second boundary that is parallel to the first, i.e. $y = a$ that also has a Dirichlet or Neumann boundary condition. The domain of the Poisson equation is now $0 < y < a$. Denote as u^1 the solution that satisfies the BC at $y = 0$. A solution that satisfies the Dirichlet or Neumann boundary conditions at $y = a$ are as follows.
Equation:

$$u(x, y, z) = u^1(x, y, z) - u^1(x, 2a - y, z), \quad \text{for } u = 0 \text{ at } y = a$$
$$u(x, y, z) = u^1(x, y, z) + u^1(x, 2a - y, z), \quad \text{for } du/dy = 0 \text{ at } y = a$$

This solution satisfies the solution at $y = a$, but no longer satisfies the solution at $y = 0$. Denote this solution as u^2 and find the solution to satisfy the BC at $y = 0$. By continuing this operation, one obtains by induction a series solution that satisfies both boundary conditions. It may be more convenient to place the boundaries symmetric with respect to the axis in order to simplify the recursion formula.

Assignment 7.3
Calculate the solution for a unit line source at the origin of the x,y plane with zero flux boundary conditions at $y = +1$ and $y = -1$. Prepare a contour plot of the solution for $0 < x < 5$. What is the limiting solution for large x? Note: The boundary conditions are conditions on the derivative. Thus the solution is arbitrary by a constant.

Existence and Uniqueness of the Solution to the Poisson Equation

If the boundary conditions for Poisson equation are the Neumann boundary conditions, there are conditions for the existence to the solution and the solution is not unique. This is illustrated as follows.
Equation:

$$\begin{aligned}
&\nabla^2 u = -\rho \quad \text{in } V, \qquad \mathbf{n} \bullet \nabla u = f \text{ on } S \\
&\iiint \nabla^2 u \, dV = -\iiint \rho \, dV \\
&\oint \mathbf{n} \nabla u \, dS = -\iiint \rho \, dV \\
&\oint f \, dS = -\iiint \rho \, dV
\end{aligned}$$

This necessary condition for the existence of a solution is equivalent to the statement that the flux leaving the system must equal the sum of sources in the system. The solution to the Poisson equation with the Neumann boundary condition is arbitrary by a constant. If a constant is added to a solution, this new solution will still satisfy the Poisson equation and the Neumann boundary condition.

Green's function for the diffusion equation

We showed above how the solution to the Poisson equation with homogeneous boundary conditions could be obtained from the Green's function by convolution and method of images. Here we will obtain the Green's function for the diffusion equation for an infinite domain in one, two, or three dimensions. The Green's function is for the parabolic PDE
Equation:

$$\nabla^2 u - a^2 \frac{\partial u}{\partial t} = -\rho\left(\mathbf{x}, t\right)$$

where the parameter a^2 represents the ratio of the storage capacity and the conductivity of the system and ρ is a known distribution of sources in space and time. The infinite domain Green's function $g_n\left(x, tx_o, t_o\right)$ is a solution of the following equation
Equation:

$$\nabla^2 g_n(\mathbf{x}, t|\mathbf{x}_o, t_o) - a^2 \frac{\partial g_n(\mathbf{x}, t|\mathbf{x}_o, t_o)}{\partial t} = -\delta\left(\mathbf{x} - \mathbf{x}_o\right)\delta\left(t - t_o\right)$$

The source term is an impulse in the spatial and time variables. The form of the Green's function for the infinite domain, for n dimensions, is (Morse and Feshbach, 1953)
Equation:

$$g_n(\mathbf{x}, t|\mathbf{x}_o, t_o) = g_n\left(R, \tau\right) = \begin{cases} \frac{1}{a^2}\left(\frac{a}{2\sqrt{\pi\tau}}\right)^n e^{-\left(a^2R^2/4\tau\right)}, & \tau > 0 \\ 0 \quad , & \tau < 0 \end{cases}$$

where
$\tau = t - t_o$
$R = |\mathbf{x} - \mathbf{x}_o|$

This Green's function satisfies an important integral property that is valid for all values of n:
Equation:

$$\int g_n\left(R, \tau\right) dV_n = \frac{1}{a^2}, \quad \tau > 0$$

This expression is an expression of the conservation of heat energy. At a time t_o at $\mathbf{x}_o$, a source of heat is introduced. The heat diffuses out through the medium, but in such a fashion that the total heat energy is unchanged.

The properties of this Green's function can be more easily seen be expressing it in a standard form
Equation:

$$a^2 g_n\left(R,\tau\right)=\begin{cases}\left(\frac{1}{\sqrt{2\pi\left(2\tau/a^2\right)}}\right)^n e^{-\left[R^2/2\left(2\tau/a^2\right)\right]}, & \tau>0\\ 0\quad, & \tau<0\end{cases}$$

The normalized function $a^2 g_n$ for $n=3$ represent the probability distribution of the location of a Brownian particle that was at x_o at time t_o. The cumulative probability is equal to unity.

The same normalized function for $n=1$, corresponds to the normal or Gaussian distribution with the standard deviation given by
Equation:

$$\sigma=\frac{\sqrt{2\tau}}{a}$$

Observe the Green's function in one, two, and three dimensions by executing greens.m and the function, greenf.m in the diffuse subdirectory of CENG501. You may wish to use the code as a template for future assignments.

<u>Step Response Function</u> The infinite domain Green's function is the *impulse response function* in space and time. The response for a distribution of sources in space or as an arbitrary function of time can be determined by convolution. In particular the response to a constant source for $\tau>0$ is the *step response function*. It has classical solutions in one and two dimensions. The unit step function or Heaviside function is the integral of the Dirac delta function.
Equation:

$$\int_{-\infty}^{t}\delta\left(t'-t_o\right)dt'=S\left(t-t_o\right)=\begin{cases}1, & t-t_o>0\\ 0, & t-t_o<0\end{cases}$$

The response function to a unit step in the source can be determined by integrating the Greens function or the impulse response function in time.
Equation:

$$\int_{-\infty}^{t} \left[\nabla^2 g_n - a^2 \frac{\partial g_n}{\partial t} \right] dt' = -\,\delta\left(\mathbf{x} - \mathbf{x}_o\right) \int_{-\infty}^{t} \delta\left(t' - t_o\right) dt'$$

$$\nabla^2 \left(\int_{-\infty}^{t} g_n \, dt' \right) - a^2 \frac{\partial \left(\int_{-\infty}^{t} g_n \, dt' \right)}{\partial t} = -\,\delta\left(\mathbf{x} - \mathbf{x}_o\right) S(t - t_o)$$

$$\nabla^2 U_n - a^2 \frac{\partial U_n}{\partial t} = -\,\delta\left(\mathbf{x} - \mathbf{x}_o\right) S(t - t_o)$$

where

$$U_n = \int_{-\infty}^{t} g_n \, dt'$$

In one dimension, the step response function that has a unit flux at $x = 0$ is (R. V. Churchill, Operational Mathematics, 1958) (note: source is $2\delta(x - 0)\, S(t - 0)$)

Equation:

$$U_1^{\text{flux}=-1} = 2a^2 \sqrt{\frac{t}{\pi\, a^2}}\, e^{\frac{-x^2}{4t/a^2}} - a^2 |x| \ \text{erfc}\left(\frac{|x|}{\sqrt{4t/a^2}} \right), \quad t > 0$$

For comparison, the function that has a value of unity at $x = 0$ (Dirichlet boundary condition) is

Equation:

$$U_1^{=1} = \text{erfc}\left(\frac{|x|}{2\sqrt{t/a^2}} \right), \quad t > 0$$

In two dimensions, the unit step response function for a *continuous line source* is (H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, 1959)

Equation:

$$\begin{aligned} U_2 &= \frac{-a^2}{4\pi}\, Ei\left(\frac{-R^2}{4t/a^2} \right), \quad t > 0 \\ R^2 &= \left(x - x_o\right)^2 + \left(y - y_o\right)^2 \\ -Ei\left(-x\right) &= \int_x^{\infty} \frac{e^{-u}}{u}\, du \\ &= \text{expint}\ (x), \quad \text{MATLAB function} \end{aligned}$$

For large times this function can be expressed as

Equation:

$$U_2^{\text{approx.}} = \frac{a^2}{4\pi} \ln\left(\frac{4t/a^2}{R^2}\right) - \frac{\gamma\, a^2}{4\pi}, \quad \text{for } \frac{4t}{a^2\, R^2} > 100$$
$$\gamma = 0.5772...$$

In three dimensions, the unit step response function for a continuous point source is (H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, 1959)
Equation:

$$U_3 = \frac{a^2}{4\pi\, R} erfc\left(\frac{R}{\sqrt{4t/a^2}}\right), \qquad t > 0$$
$$R = \left[(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right]^{1/2}$$

Note: The a^2 .factor has the units of time/L^2. If time is made dimensionless with respect to a^2/R_o^2 and R with respect to R_o , then the factor will disappear from the argument of the erfc.

Assignment 7.4
Plot the profiles of the response to a continuous source in 1, 2, and 3 dimensions using the MATLAB code *contins.m* and *continf.m* in the *diffuse* subdirectory. From the integral of the profiles as a function of time, determine the magnitude, spatial and time dependence of the source. Note: The exponential integral function, *expint* will give error messages for extreme values of the argument. It still computes the correct values of the function.

Convective-Diffusion Equation

The convective-diffusion equation in one dimension will be expressed in terms of velocity and dispersion,
Equation:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = K\frac{\partial^2 u}{\partial x^2}$$
$$u(x,) = 0, \quad x > 0$$
$$u(0,t) = 1, \quad t > 0$$

The independent variables can be transformed from (x, t) to a spatial coordinate that translates with the velocity of the wave in the absence of dispersion, (y, t).
Equation:

$$y = x - v\,t$$

This transforms the equation to the diffusion equation in the transformed coordinates.
Equation:

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial y^2}$$

To see this, we will transform the differentials from x to y.
Equation:

$$\frac{\partial y}{\partial t} = -v$$
$$\frac{\partial y}{\partial x} = 1$$

The total differentials expressed as a function of (x, t) or (y, t) are equal to each other.
Equation:

$$du = \left(\frac{\partial u}{\partial t}\right)_x dt + \left(\frac{\partial u}{\partial x}\right)_t dx$$
$$du = \left(\frac{\partial u}{\partial t}\right)_y dt + \left(\frac{\partial u}{\partial y}\right)_t dy$$

The total differentials expressed either way are equal. The partial derivatives in t and x can be expressed in terms of partial derivatives in t and y by equating the total differentials with either dt or dx equal to zero and dividing by the non-zero differential.
Equation:

$$
\begin{aligned}
\left(\frac{\partial u}{\partial t}\right)_x &= \left(\frac{\partial u}{\partial t}\right)_y + \left(\frac{\partial u}{\partial y}\right)_t \left(\frac{\partial y}{\partial t}\right)_x \\
&= \left(\frac{\partial u}{\partial t}\right)_y - v\left(\frac{\partial u}{\partial y}\right)_t \\
& \quad \text{and} \\
\left(\frac{\partial u}{\partial x}\right)_t &= \left(\frac{\partial u}{\partial y}\right)_t \left(\frac{\partial y}{\partial x}\right)_t \\
&= \left(\frac{\partial u}{\partial y}\right)_t \\
\left(\frac{\partial^2 u}{\partial x^2}\right)_t &= \left(\frac{\partial^2 u}{\partial y^2}\right)_t
\end{aligned}
$$

Substitution into the original equation results in the transformed equation. This result could have been derived in fewer steps by using the chain rule but would not have been as enlightening.

The boundary condition at $x = 0$ is now at changing values of y. We will seek an approximate solution that has the boundary condition $u(y \to -\infty) = 1$. A simple solution can be found for the following initial and boundary conditions.
Equation:

$$
\begin{aligned}
& u\left(y, 0\right) = \begin{cases} 1, & y < 0 \\ 1/2, & y = 0 \\ 0, & y > 0 \end{cases} \\
& u(y \to -\infty, t) = 1 \\
& u(y \to \infty, t) = 0
\end{aligned}
$$

This system is a step with no dispersion at $t = 0$. Dispersion occurs for $t > 0$ as the wave propagates through the system. The solution can be found with a similarity transform, which we will discuss later. For now, the approximate solution is given as
Equation:

$$
u = \tfrac{1}{2}\,\mathrm{erfc}\left(\frac{y}{\sqrt{4K\,t}}\right) = \tfrac{1}{2}\,\mathrm{erfc}\left(\frac{x - v\,t}{\sqrt{4K\,t}}\right)
$$

The boundary condition at $x = 0$ will be approximately satisfied after a small time unless the Peclet number is very small.

Similarity transformation

In some cases a partial differential equation and its boundary conditions (and initial condition) can be transformed to an ordinary differential equation with boundary conditions by combining two independent variables into a single independent variable. We will illustrate the approach here with the diffusion equation. It will be used later for hyperbolic PDEs and for the boundary layer problems.

The method will be illustrated for the solution of the one-dimensional diffusion equation with the following initial and boundary conditions. The approach will follow that of the Hellums-Churchill method.
Equation:

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = K \tfrac{\partial^2 u}{\partial x^2}, \quad t > 0,\ x > 0 \\ &u\left(x, 0\right) = u_{IC} \\ &u\left(0, t\right) = u_{BC} \end{aligned}$$

The PDE, IC and BC are made dimensionless with respect to reference quantities.
Equation:

$$\begin{aligned} u^* &= \tfrac{u - u_{IC}}{u_o} \\ t^* &= \tfrac{t}{t_o} \\ x^* &= \tfrac{x}{x_o} \\ \tfrac{\partial u^*}{\partial t^*} &= \left[\tfrac{K\, t_o}{x_o^2}\right] \tfrac{\partial^2 u^*}{\partial x^{*2}} \\ u^*(x^*, 0) &= 0 \\ u^*(0, t^*) &= \left[\tfrac{u_{BC} - u_{IC}}{u_o}\right] = 1, \quad \Rightarrow u_o = u_{BC} - u_{IC} \end{aligned}$$

There are three unspecified reference quantities and two dimensionless groups. The BC can be specified to equal unity. However, the system does not have a characteristic time or length scales to specify the dimensionless group in the PDE. This suggests that the system is over specified and the independent variables can be combined to specify the dimensionless group in the PDE to equal 1/4.
Equation:

$$\begin{aligned} &\left[\tfrac{K\, t_o}{x_o^2}\right] = \tfrac{1}{4} \quad \Rightarrow \eta = \tfrac{x}{\sqrt{4K\, t}} \quad \text{is dimensionless} \\ &u(x, t) = u(\eta) \end{aligned}$$

The partial derivatives will now be expressed as a function of the derivatives of the transformed similarity variable.

Equation:

$$
\begin{aligned}
\frac{\partial \eta}{\partial x} &= \frac{1}{\sqrt{4K\,t}} \\
\frac{\partial \eta}{\partial t} &= \frac{-\eta}{2t} \\
\frac{\partial u}{\partial t} &= \frac{du}{d\eta}\,\frac{\partial \eta}{\partial t} = \frac{-\eta}{2t}\,\frac{du}{d\eta} \\
\frac{\partial u}{\partial x} &= \frac{1}{\sqrt{4K\,t}}\,\frac{du}{d\eta} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{4K\,t}\,\frac{d^2u}{d\eta^2}
\end{aligned}
$$

The PDE is now transformed into an ODE with two boundary conditions.

Equation:

$$
\begin{aligned}
&\frac{d^2u^*}{d\eta^2} + 2\eta\,\frac{du^*}{d\eta} = 0 \\
&u^*(\eta = 0) = 1 \\
&u^*(\eta \to \infty) = 0
\end{aligned}
$$

Equation:

$$
\begin{aligned}
&\text{Let} \qquad v = \frac{du^*}{d\eta} \\
&\frac{dv}{d\eta} + 2\eta\,v = 0 \\
&\frac{dv}{v} = d\ \ln\ v = -2\eta\,d\eta \\
&v = C_1\,e^{-\eta^2} \\
&u^* = C_1\ \int_0^\eta e^{-\eta^2}\,d\eta + C_2 \\
&u^*(\eta = 0) = 1 \quad \Rightarrow \quad C_2 = 1 \\
&u^*(\eta \to \infty) = 0 = C_1\ \int_0^\infty e^{-\eta^2}\,d\eta + 1 \\
&C_1 = \frac{-1}{\int_0^\infty e^{-\eta^2}\,d\eta} \\
&u^*(\eta) = \frac{-\int_0^\eta e^{-\eta^2}\,d\eta}{\int_0^\infty e^{-\eta^2}\,d\eta} + 1 = \operatorname{erfc}(\eta) \\
&u^*(x,t) = \operatorname{erfc}\left(\frac{x}{\sqrt{4K\,t}}\right)
\end{aligned}
$$

Therefore, we have a solution in terms of the combined similarity variable that is a solution of the PDE, BC, and IC.

Complex potential for irrotational flow

Incompressible, irrotational flows in two dimensions can be easily solved in two dimensions by the process of conformal mapping in the complex plane. First we will review the kinematic conditions that lead to the PDE and boundary conditions. Because the flow is irrotational, the velocity is the gradient of a velocity potential. Because the flow is solenoidal, the velocity is also the curl of a vector potential. Because the flow is two dimensional, the vector potential has only one non-zero component that is identified as the stream function. The kinematic condition at solid boundaries is that the normal component of velocity is zero. No condition is placed on the tangential component of velocity at solid surfaces because the fluid must be inviscid in order to be irrotational.

Equation:

$$
\begin{aligned}
\mathbf{v} &= \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, 0\right) = (v_x, v_y, v_z) \\
\nabla \bullet \mathbf{v} &= 0 = \nabla^2\phi \\
\mathbf{v} &= \nabla \times \mathbf{A} = \left(\frac{\partial A_3}{\partial y},\ \frac{-\partial A_3}{\partial x},\ 0\right) \\
&= \left(\frac{\partial\psi}{\partial y},\ \frac{-\partial\psi}{\partial x},\ 0\right) = (v_x, v_y, v_z) \\
\nabla \times \mathbf{v} &= w = 0 = \nabla^2\psi \\
\mathbf{v} &= \nabla\phi = \nabla \times \mathbf{A} \\
\left(\frac{\partial\phi}{\partial x},\ \frac{\partial\phi}{\partial y}, 0\right) &= \left(\frac{\partial\psi}{\partial y},\ \frac{-\partial\psi}{\partial x},\ 0\right) \\
\text{or} \quad \frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y} \\
\text{and} \quad \frac{\partial\phi}{\partial y} &= \frac{-\partial\psi}{\partial x}
\end{aligned}
$$

Both functions are a solution of the Laplace equation, i.e., they are harmonic and the last pair of equations corresponds to the Cauchy-Riemann conditions if ϕ and ψ are the real and imaginary conjugate parts of a complex function, $w(z)$.

Equation:

$$
\begin{aligned}
w(z) &= \phi(z) + i\,\psi(z) \\
z &= x + i\,y \\
&= r\,e^{i\theta} = r\,(\cos\theta + i\ \sin\theta), \quad r = |z| = \left(x^2 + y^2\right)^{1/2}, \quad \theta = \arctan\,(y/x) \\
&\text{or} \\
\phi(z) &= \text{real}[w(z)] \\
\psi(z) &= \text{imaginary}[w(z)]
\end{aligned}
$$

The Cauchy-Riemann conditions are the necessary and sufficient condition for the derivative of a complex function to exist at a point z_o , i.e., for it to be *analytical*. The necessary condition can be illustrated by equating the derivative of $w(z)$ taken along the real and imaginary axis.

Equation:

$$
\begin{aligned}
\frac{dw(z)}{dz} &= \text{Re}(w'\,(z)) + i\,\text{Im}(w'\,(z)) \\
&= \lim\ \frac{\delta\phi + i\delta\psi}{\delta x + i0} = \frac{\partial\phi}{\partial x} + i\,\frac{\partial\psi}{\partial x} \\
&= \lim\ \frac{\delta\phi + i\delta\psi}{0 + i\delta y}\,\frac{i}{i} = \frac{\partial\psi}{\partial y} - i\,\frac{\partial\phi}{\partial y} \\
&\therefore\ \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \\
\text{and}\ \frac{\partial\psi}{\partial x} &= -\frac{\partial\phi}{\partial y}, \quad \text{Q.E.D.}
\end{aligned}
$$

Also, if the functions have second derivatives, the Cauchy-Riemann conditions imply that each function satisfies the Laplace equation.

Equation:

$$
\begin{aligned}
\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= 0 \\
\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} &= 0
\end{aligned}
$$

The Cauchy-Riemann conditions also imply that the gradient of the velocity potential and the stream function are orthogonal.

Equation:

$$
(\nabla\phi) \bullet (\nabla\psi) = \frac{\partial\phi}{\partial x}\,\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\,\frac{\partial\psi}{\partial y} = 0
$$

If the gradients are orthogonal then the equipotential lines and the streamlines are also orthogonal, with the exception of stagnation points where the velocity is zero.

Since the derivative
Equation:

$$\frac{dw}{dz} = \lim_{|\delta z \to 0|} \frac{\delta\, w}{\delta\, z}$$

is independent of the direction of the differential δz in the (x, y) -plane, we may imagine the limit to be taken with δz remaining parallel to the x-axis $(\delta z = \delta x)$ giving
Equation:

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\,\frac{\partial \psi}{\partial x} = v_x - i\, v_y$$

Now choosing z to be parallel to the y-axis $(\delta z = i\delta y)$,
Equation:

$$\frac{dw}{dz} = \frac{1}{i}\,\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} = -i\, v_y + v_x = v_x - i\, v_y$$

These equations are a restatement that an analytical function has a derivative defined in the complex plane. Moreover, we see that the real part of $w'\,(z)$ is equal to v_x and the imaginary part of $w'\,(z)$ is equal to $-v_y$. If v is written for the magnitude of **v** and for the angle between the direction of **v** and the x-axis, the expression for dw/dz becomes
Equation:

$$\begin{aligned}
&\tfrac{dw}{dz} = v_x - i\, v_y = v\, e^{-i\theta} \\
&\text{or} \\
&v_x = \text{real}\big[\tfrac{dw}{dz}\big] \\
&v_y = -\,\text{imaginary}\big[\tfrac{dw}{dz}\big]
\end{aligned}$$

<u>Flow Fields.</u> The simplest flow field that we can imagine is just a constant translation, $w = (U - iV)z$ where U and V are real constants. The components of the velocity vector can be determined from the differential.
Equation:

$$w(z) = (U - iV)z = (U - iV)(x + i\,y) = Ux + Vy + i(-Vx + Uy) = \phi + i\,\psi$$
$$\frac{dw}{dz} = (U - iV) = v_x - i\,v_y$$
$$v_x = U, \quad v_y = V$$
$$\phi = Ux + Vy, \quad \psi = -Vx + Uy$$

Another simple function that is analytical with the exception at the origin is
Equation:

$$\begin{aligned} w\,(z) &= A\,z^n = A\,r^n e^{i\,n\,\theta} \\ &= A\,r^n \cos\ n\theta + i\,A\,r^n \sin\ n\theta \\ &= \phi + i\,\psi \\ & \quad \text{thus} \\ \phi &= A\,r^n \cos\ n\theta \\ \psi &= A\,r^n \sin\ n\theta \\ \frac{dw}{dz} &= n\,A\,z^{n-1} = v_x - i\,v_y \end{aligned}$$

where A and n are real constants. The boundary condition at stationary solid surfaces for irrotational flow is that the normal component of velocity is zero or the surface coincides with a streamline. The expression above for the stream function is zero for all r when $\theta = 0$ and when $\theta = \pi/n$. Thus these equations describe the flow between these boundaries are illustrated below.

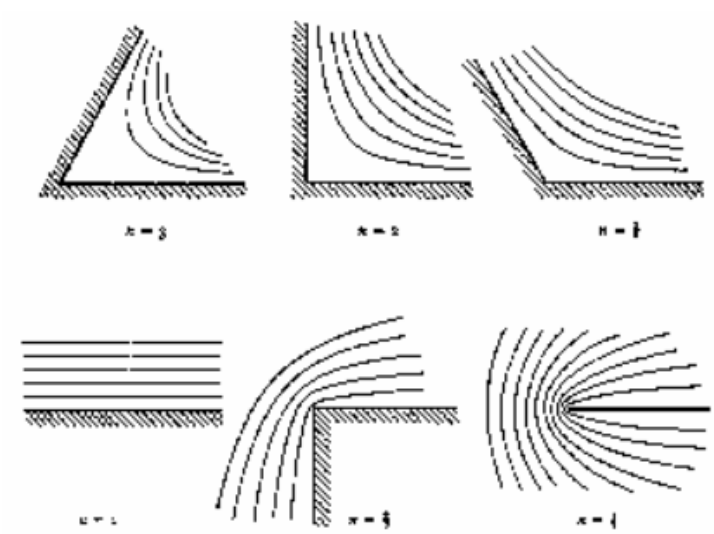


Fig. 6.5.1 (Batchelor, 1967) Irrotational flow in the region between two straight zero-flux boundaries intersecting at an angle π/n.

Earlier we discussed the Green's function solution of a line source in two dimensions. The same solution can be found in the complex domain. A function that is analytical everywhere except the singularity at z_o is the function for a line source of strength m.
Equation:

$$w(z) = \frac{m}{2\pi} \ln(z - z_o), \quad \text{line source}$$
$$\frac{\mathrm{dw}}{\mathrm{dz}} = \frac{m}{2\pi} \frac{1}{(z - z_o)} = v_x - i\, v_y$$

This results can be generalized to multiple line sources or sinks by superposition of solutions. A special case is that of a source and sink of the same magnitude.
Equation:

$$w(z) = \sum_i \frac{m_i}{2\pi} \ln(z - z_i), \quad \text{multiple line sources}$$
$$w(z) = \frac{m}{2\pi} \ln\left(\frac{z - z_o}{z + z_o}\right), \quad \text{source} - \text{sink pair}$$

The above flow fields can be viewed with the MATLAB code *corner.m, linesource.m, and multiple.m* in the *complex* subdirectory.

Assignment 7.5
Line Source Solution For z_o at the origin, derive expressions for the flow potential, stream function, components of velocity, and magnitude of velocity for the solution to the line source in terms of r and θ . Plot the flow potentials and stream functions. Compute and plot the flow potentials and stream function for the superposition of multiple line sources corresponding to the zero flux boundary conditions at $y = +1$ and -1 of the earlier assignment.

<u>The circle theorem.</u> (Batchelor, 1967) The following result, known as the circle theorem (Milne-Thompson, 1940) concerns the complex potential representing the motion of an inviscid fluid of infinite extent in the presence of a single internal boundary of circular form. Suppose first that in the absence of the circular cylinder the complex potential is
Equation:

$$w^o = f(z)$$

and that $f(z)$ is free from singularities in the region $|z| \leq a$, where a is a real length. If now a stationary circular cylinder of radius a and center at the origin bounds the fluid internally, the flow is modified to the following complex potential:
Equation:

$$w^1 = f(z) + \overline{f}\left(a^2/z\right)$$

We show that the surface of the cylinder, $|z| = a$, is a streamline.
Equation:

$$a^2 = z\,\overline{z}$$

Equation:

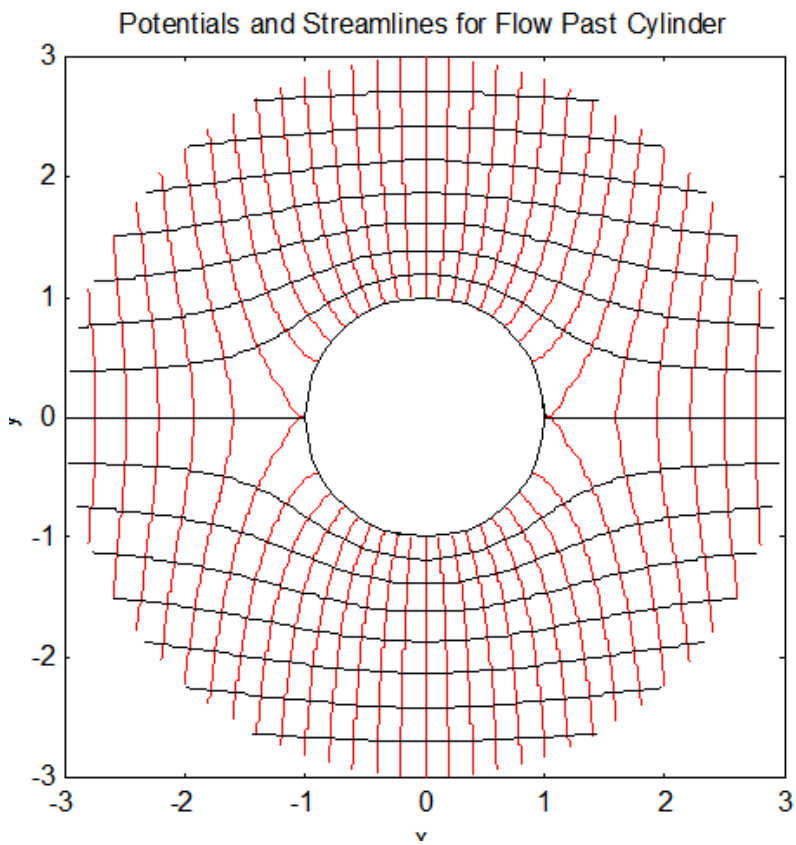
$$\begin{aligned} w^1(z)\big|_{|z|=a} &= f(z)\big|_{|z|=a} + \overline{f}\,\left(a^2/z\right)\Big|_{|z|=a} \\ &= f(z)\big|_{|z|=a} + \overline{f}\,\left(z\overline{z}/z\right)\Big|_{|z|=a} \\ &= f(z)\big|_{|z|=a} + \overline{f}\,\left(\overline{z}\right)\Big|_{|z|=a} \\ &= 2\ \mathrm{Real}\Big[f(z)\big|_{|z|=a}\Big] + i\,0 \\ &= \phi_{|z|=a} + i\ \psi_{|z|=a} \end{aligned}$$

A complex potential of this form thus has $|z| = a$ as a streamline, $\psi = 0$; and it has the same singularities outside $|z| = a$ as $f(z)$, since if z lies outside $|z| = a$, a^2/z lies in the region inside this circle where $f(z)$ is known to be free from singularities. Consequently the additional term $\overline{f}\,\left(a^2/z\right)$ in the equation represents fully the modification to the complex potential due to the presence of the circular cylinder. It should be noted that the complex potentials considered, both in the absence and in the presence of the circular cylinder, refer to the flow relative to axis such that the cylinder is stationary.

The simplest possible application of the circle theorem is to the case of a circular cylinder held fixed in a stream whose velocity at infinity is uniform with components $(-U, -V)$. In the absence of the cylinder the complex potential is $-(U - i\,V)z$, it is singular at infinity and the circle theorem shows that, with the cylinder present,
Equation:

$$w\,(z) = -\,(U - i\,V)z - (U + i\,V)\,a^2/z$$

The potentials and streamlines for the steady translation of an inviscid fluid past a circular cylinder can be viewed with the MATLAB code *circle.m*.



<u>Conformal Transformation (Batchelor, 1967).</u> We now have the complex potential flow solutions of several problems with fairly simple boundary conditions. These solutions are analytical functions whose real and imaginary parts satisfy the Laplace equation. They also have a streamline that coincides with the boundary to satisfy the condition of zero flux across the boundary. Conformal transformations can be used to obtain solutions for boundaries that are transformed to different shapes. Suppose we have an analytical function $w(z)$ in the $z = x + iy$ plane. This solution can be transformed to the $\zeta = \xi + i\eta$ plane as another analytical function provided that the relation between these two planes, $\zeta = F(z)$ is an analytical function. This mapping is a connection between the shape of a curve in the z - plane and the shape of the curve traced out by the corresponding set of points in the $\zeta - plane$. The solution in the $\zeta - plane$ is analytical, i.e., its derivative defined, because the mapping, $\zeta = F(z)$ is an analytical function. The inverse transformation is also analytical.

Equation:

$$\frac{dw(\varsigma)}{d\varsigma} = \frac{d\,w(z)}{dz}\,\frac{dz}{d\varsigma} = \frac{\frac{d\,w(z)}{dz}}{\frac{dF(z)}{dz}}$$
$$\frac{d\,w(z)}{dz} = \frac{dw(\varsigma)}{d\varsigma}\,\frac{dF(z)}{dz}$$

$w(\zeta)$ is thus the complex potential of an irrotational flow in a certain region of the ζ-plane, and the flow in the z-plane is said to been 'transformed' into flow in the ζ-plane. The family of equipotential lines and streamlines in the z-plane given by $\phi(x,y) = \text{const.}$ and $\psi(x,y) = \text{const.}$ transform into families of curves in the ζ-plane on which ϕ and ψ are constant and which are equipotential lines and streamlines in the ζ-plane. The two families are orthogonal in their respective plane, except at singular points of the transformation. The velocity components at a point of the flow in the ζ-plane are given by

Equation:

$$v_\xi - iv_\eta = \frac{dw}{d\varsigma} = \frac{dw}{dz}\frac{dz}{d\varsigma} = (v_x - iv_y)\frac{dz}{d\varsigma}$$

This shows that the magnitude of the velocity is changed, in the transformation from the z-plane to the ζ-plane, by the reciprocal of the factor by which linear dimensions of small figures are changed. Thus the kinetic energy of the fluid contained within a closed curve in the z-plane is equal to the kinetic energy of the corresponding flow in the region enclosed by the corresponding in the ζ-plane.

<u>Flow around elliptic cylinder (Batchelor, 1967).</u> The transformation of the region outside of an ellipse in the z-plane into the region outside a circle in the ζ-plane is given by

Equation:

$$z = \varsigma + \frac{\lambda^2}{\varsigma}$$
$$\varsigma = \frac{1}{2}z + \frac{1}{2}\left(z^2 - 4\lambda^2\right)^{1/2}$$

where λ is a real constant so that

Equation:

$$x = \xi\left(1 + \frac{\lambda^2}{|\varsigma|^2}\right), \quad y = \eta\left(1 - \frac{\lambda^2}{|\varsigma|^2}\right)$$

This converts a circle of radius c with center at the origin in the - plane into the ellipse

Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the z-plane, where
Equation:

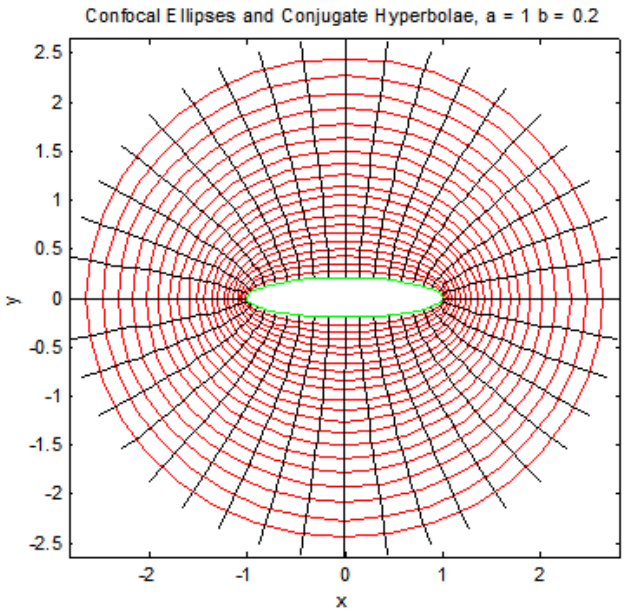
$$\lambda = \frac{1}{2}\left(a^2 - b^2\right)^{1/2}$$

If the ellipse is mapped into a circle in the ζ-plane, it is convenient to use polar coordinates (r, θ), especially since the boundary corresponds to a constant radius. The radius that maps to the elliptical boundary is (*ellipse.m* in the *complex* directory)
Equation:

$$r_o = \frac{1}{2}\,\log\left(\frac{a+b}{a-b}\right)$$

The transformation from the polar coordinates to the z - plane is defined by
Equation:

$$z = 2\lambda\ \cosh\ \omega$$
$$\text{where}$$
$$\omega = r + i\,\theta$$



Transformation of cylindrical polar coordinates into orthogonal, elliptical coordinates

The polar coordinates (r, θ), transform to an orthogonal set of curves which are confocal ellipses and conjugate hyperbolae.

This transformation can be substituted into the complex potential expression for the flow of a fluid past a circular cylinder.
Equation:

$$w = -\frac{1}{2}(a+b)\left[(U-iV)e^{\omega - r_o} + (U+iV)e^{r_o - \omega}\right]$$

It is convenient to write - for the angle which the direction of motion of the flow at infinity makes with the x - axis so that
Equation:

$$U + iV = \left(U^2 + V^2\right)^{1/2} e^{-i\alpha}$$

The complex potential now becomes

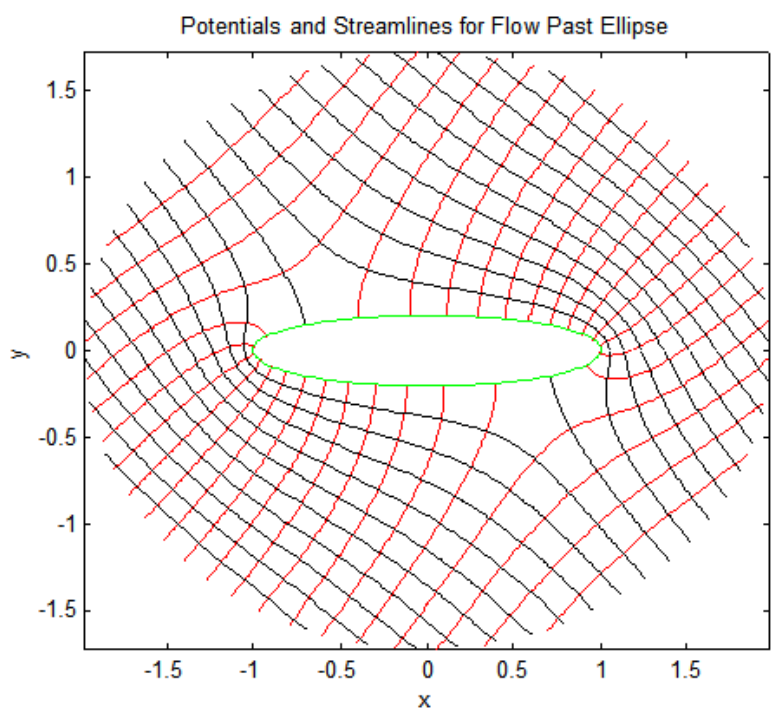
Equation:

$$w = -\left(U^2 + V^2\right)^{1/2}(a+b)\,\cosh\,(\omega - r_o + i\alpha)$$

The corresponding velocity potential and stream function are
Equation:

$$\phi = -\left(U^2 + V^2\right)^{1/2}(a+b)\;\cosh\,(r - r_o)\,\cos\,(\theta + \alpha)$$
$$\psi = -\left(U^2 + V^2\right)^{1/2}(a+b)\,\sinh\,(r - r_o)\,\sin\,(\theta + \alpha)$$

The velocity potentials and streamlines are illustrated below for flow past an elliptical cylinder (*fellipse.m* in the *complex* directory). Note the stagnation streamlines on either side of the body. These two stagnation points are regions of maximum pressure and result in a torque on the body. Which way will it rotate?



Flow past an ellipse of an inviscid fluid that is in steady translation at infinity.

Pressure distribution. When an object is in a flow field, one may wish to determine the force exerted by the fluid on the object, or the 'drag' on the object. Since the flow field discussed here has assumed an inviscid fluid, it is not possible to determine the viscous drag or skin friction directly from the flow field. It is possible to determine the 'form drag' from the normal stress or pressure distribution around the object. However, one must be critical to determine if the calculated flow field is physically realistic or if some important phenomena such as boundary layer separation may occur but is not allowed in the complex potential solution.

The Bernoulli theorems give the relation between the magnitude of velocity and pressure. We have assumed irrotational, incompressible flow. If in addition we assume the body force can be neglected, then the quantity, H, must be constant along a streamline.
Equation:

$$H = \frac{p}{\rho} + \frac{1}{2} v^2 = \text{constant}$$
$$p = -\frac{1}{2} \rho\, v^2 + \text{constant}, \quad \text{since } \rho \text{ is also constant}$$

The pressure relative to some datum can be determined by the square of the magnitude of velocity. This is easily calculated from the complex potential.
Equation:

$$\frac{dw}{dz} = v_x - i\, v_y$$
$$\overline{\frac{dw}{dz}} = v_x + i\, v_y$$
$$\text{thus}$$
$$\frac{dw}{dz}\, \overline{\frac{dw}{dz}} = v_x^2 + v_y^2 = v^2$$

There are some theorems that facilitate the integration of pressure around bodies in the complex plane, but they will not be discussed here. The pressure and tangential velocity profiles for the inviscid flow around an object are needed for calculation of the viscous flow in the boundary layer between the solid boundary and the inviscid outer flow.

Assignment 7.6
Pressure profiles Calculate the pressure field or the square of the velocity field for the flow in or around a corner and the flow past a circular cylinder. Look at the expression for the corner flow. Under what conditions is there a flow singularity? Show the pressure or velocity squared pseudo-color for wall angles of $\pi/2$, π, $3\pi/2$, and 2π . Which cases are physically realistic and what do you think happens in the unrealistic cases? What is special about the pressure profile around the circular cylinder? What

value of form drag will it predict? Is it realistic and if not, why not? Add the following code to the code for corner flow and flow around a circular cylinder. pause

```
%  Calculate pressure distribution from square of velocity
(your code here to calculate pressure field)
pcolor(x,y,p)
colormap(hot)
shading flat
axis image
```

Solution of hyperbolic systems

The conservation equations for material, momentum, and energy reduce to first order PDE in the absence of diffusivity, dispersion, viscosity, and heat conduction. In thin films, viscosity may be a dominant effect in the velocity profile normal to the surface but the continuity equation integrated over the film thickness will have only first order spatial derivatives unless the effects of interfacial curvature become important. In one dimension, the system of first order partial differential equations can be calculated by the method of characteristics [A. Jeffrey (1976), H.-K. Rhee, R. Aris, and N. R. Amundson (1986, 1987)]. Here we will only consider the case of a single dependent variable with constant initial and boundary conditions. Denote the dependent variable as S and the independent variables as x and t. The differential equation with its initial and boundary conditions are as follows.

Equation:

$$\begin{aligned}
&\frac{\partial S}{\partial t} + \frac{\partial f(S)}{\partial x} = 0, \quad t > 0,\ x > 0 \\
&S\left(x, 0\right) = S_{IC} \\
&f\left(0, t\right) = f_{BC}
\end{aligned}$$

The dependent variable can be normalized such that the initial condition is equal to zero and the boundary condition is equal to unity. Thus, henceforth it is assumed such a transformation has been made. The dependent variable will be called 'saturation' and the flux called 'fractional flow' to use the nomenclature for multiphase flow in porous media. However, the dependent variable could be film thickness in film drainage or height of a free surface as in water waves. The PDE, IC, and BC are rewritten as follows.

Equation:

$$\frac{\partial S}{\partial t} + \frac{df(S)}{dS}\frac{\partial S}{\partial x} = 0, \quad t > 0,\ x > 0$$
$$S(x,0) = 0$$
$$f(0,t) = 1$$

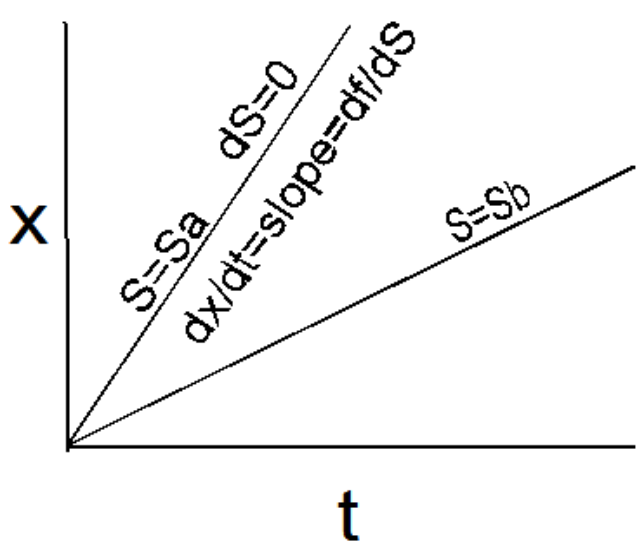
The differential, df/dS is easily calculated since there is only one independent saturation. If there were three or more phases this differential would be a Jacobian matrix. The locus of constant saturation will be sought by taking the total differential of $S(x,t)$.

Equation:

$$dS = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial x}dx = 0$$

Equation:

$$\begin{aligned}\left(\frac{dx}{dt}\right)_{dS=0} &= -\frac{\partial S/\partial t}{\partial S/\partial x} \\ &= \frac{df}{dS}(S) \\ &= v_S\end{aligned}$$



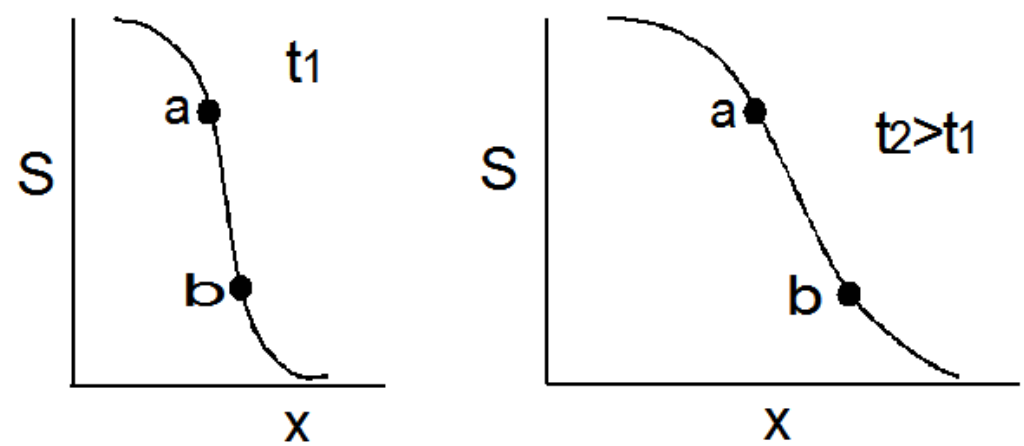
This equation expresses the velocity that a particular value of saturation propagates through the system, i.e., the *saturation velocity*, v_S is equal to the slope of the fractional

flow curve. It is also the slope of a trajectory of constant saturation (i.e., $dS = 0$) in the (x, t) space. Since we are assuming constant initial and boundary conditions, changes in saturation originate at $(x, t) = (0, 0)$. From there the changes in saturation, called waves, propagate in trajectories of constant saturation. We assume that df/dS is a function of saturation and independent of time or distance. This assumption will result in the trajectories from the origin being straight lines if the initial and boundary conditions are constant. The trajectories can easily be calculated from the equation of a straight line.

Equation:

$$x(S) = \frac{df(S)}{dS}\ t$$

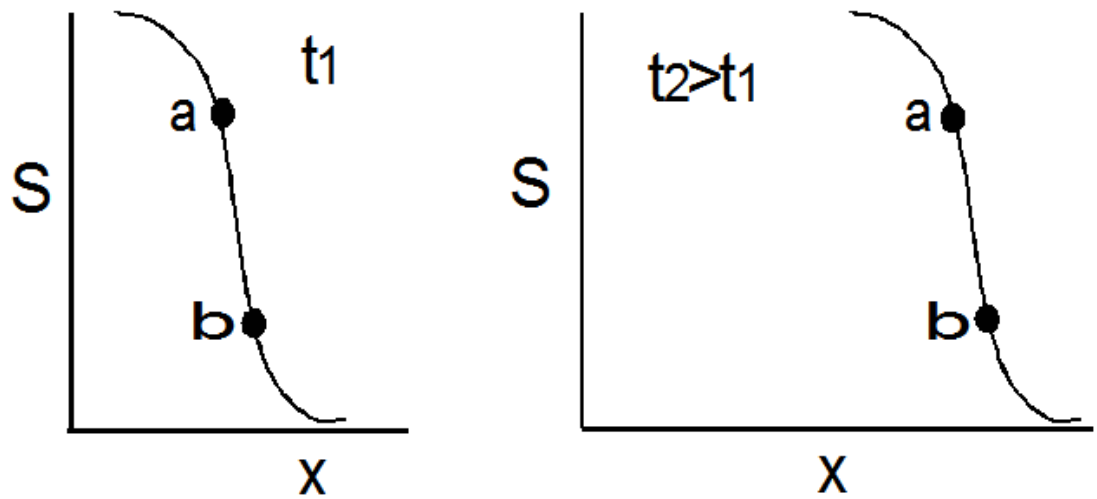
Wave: A composition (or saturation) change that propagates through the system.



Spreading wave: A wave in which neighboring composition (or saturation) values become more distant upon propagation.

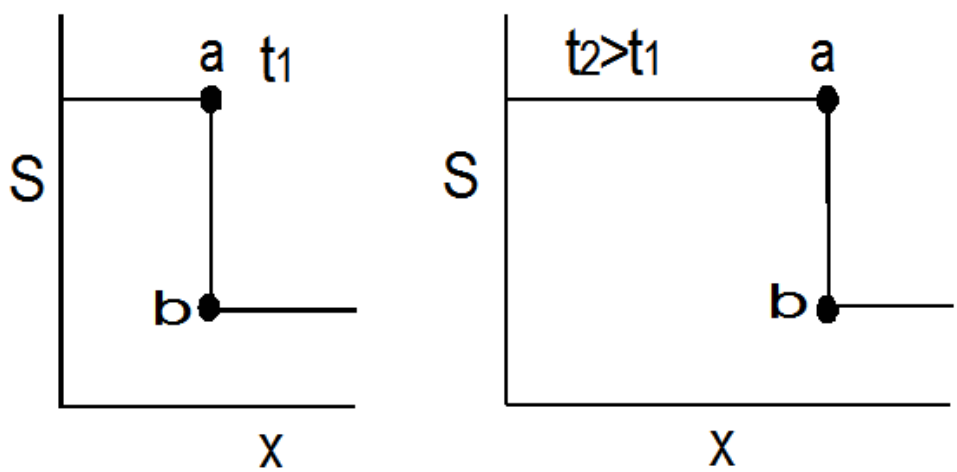
Equation:

$$\left(\frac{dx}{dt}\right)_{S_a} < \left(\frac{dx}{dt}\right)_{S_b}$$

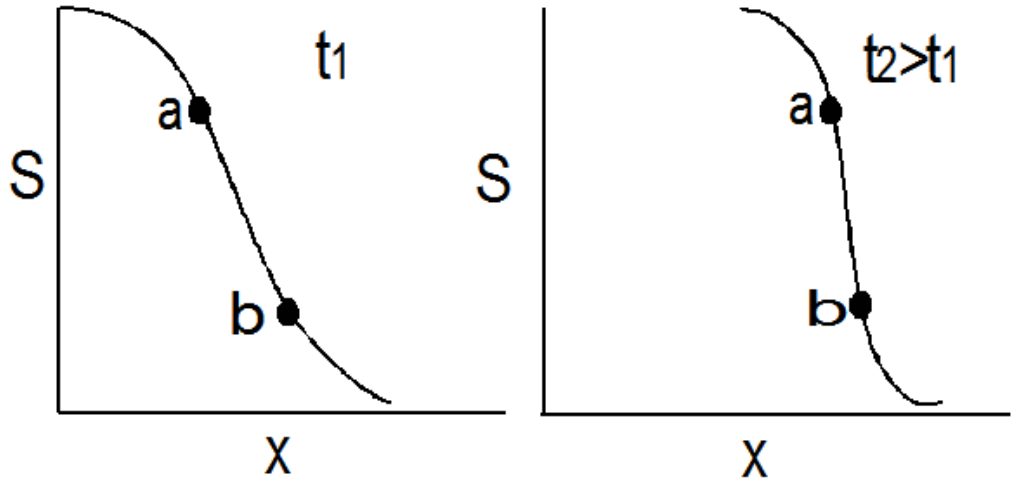


Indifferent waves: A wave in which neighboring composition (or saturation) values maintain the same relative position upon propagation.
Equation:

$$\left(\frac{dx}{dt}\right)_{S_a} = \left(\frac{dx}{dt}\right)_{S_b}$$



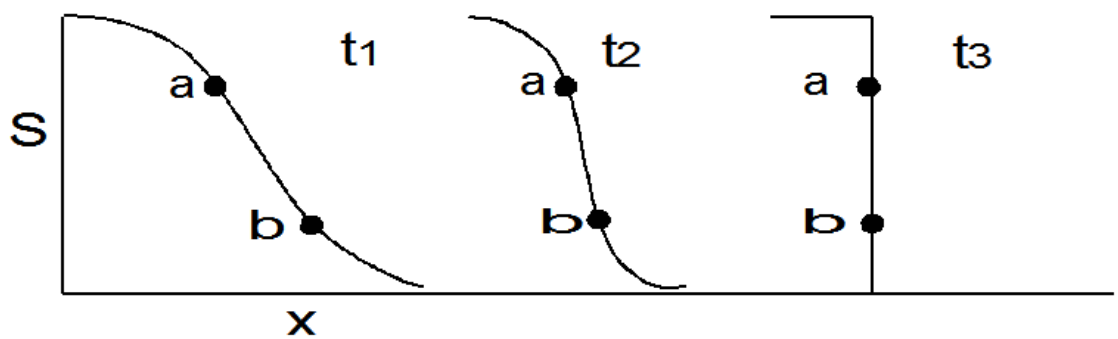
Step Wave: An indifferent wave in which the compositions change discontinuously.



Self Sharpening Waves: A wave in which neighboring compositions (saturations) become closer together upon propagation.
Equation:

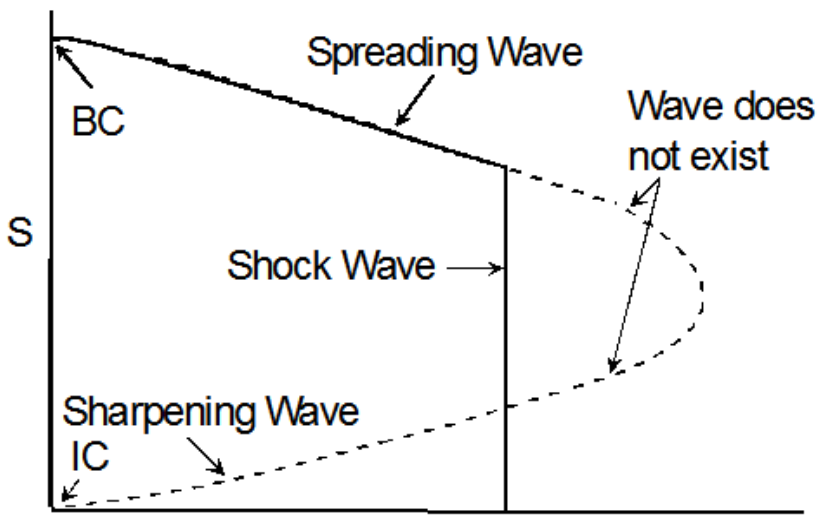
$$\left(\frac{dx}{dt}\right)_{S_a} > \left(\frac{dx}{dt}\right)_{S_b}$$

Shock Wave: A wave of composition (saturation) discontinuity that results from a self sharpening wave.



Rule: Waves originating from the same point (e.g., constant initial and boundary conditions) must have nondecreasing velocities in the direction of flow. This is another way of saying that when several waves originate at the same time, the slower waves can not be ahead of the faster waves. If slower waves from compositions close to the initial conditions originate ahead of faster waves, a shock will form as the faster waves

overtake the slower waves. This is equivalent to the statement that a sharpening wave can not originate from a point; it will immediately form a shock.

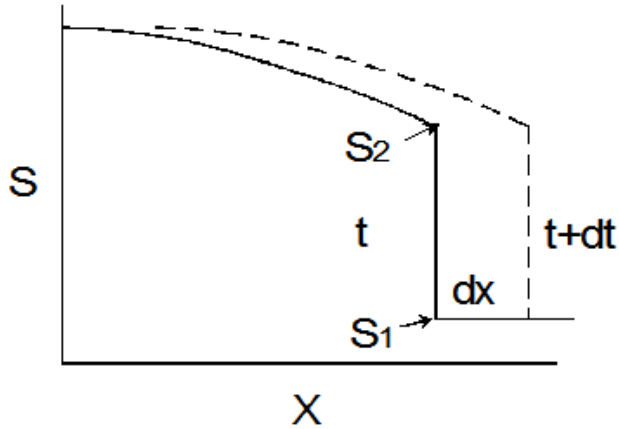


Equation:

$$\frac{x}{t} = \left(\frac{dx}{dt}\right)_{dS=0} = \frac{df}{ds}(S) = v(S)$$

Mass Balance Across Shock

We saw that sharpening wave must result in a shock but that does not tell us the velocity of a shock nor the composition (saturation) change across the shock. To determine these we must consider a mass balance across a shock. This is sometimes called an integral mass balance as opposed to the differential mass balance derived earlier for continuous composition (saturation) changes.



Equation:

$$\text{Accumulation}: \qquad \varphi\, A\Delta x(S_2 - S_1)$$

Equation:

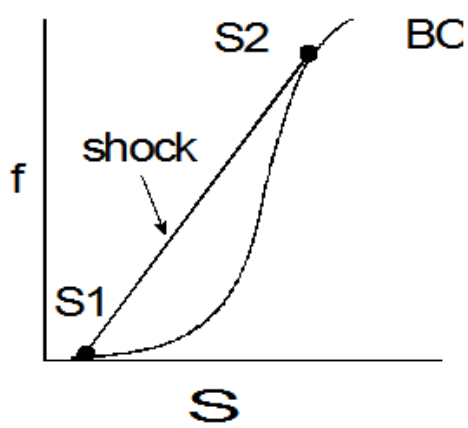
$$\text{input} - \text{output}: \qquad u\, A\Delta t(f_2 - f_1)$$

Equation:

$$\varphi\, A\Delta x\, \Delta S = u\, A\Delta t\, \Delta f$$

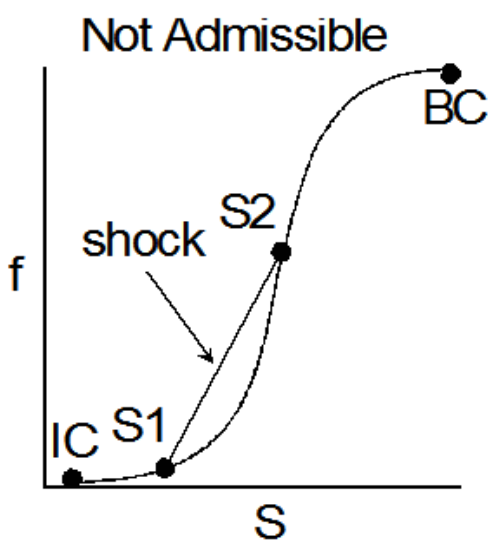
Equation:

$$\left(\frac{dx_D}{dt_D}\right)_{\Delta S} = \frac{\Delta f}{\Delta S}$$

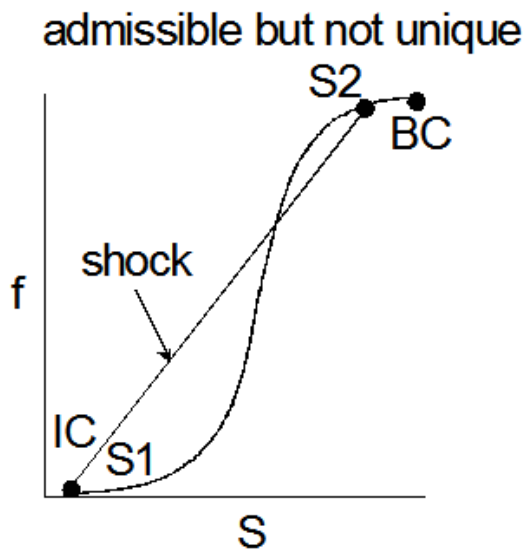


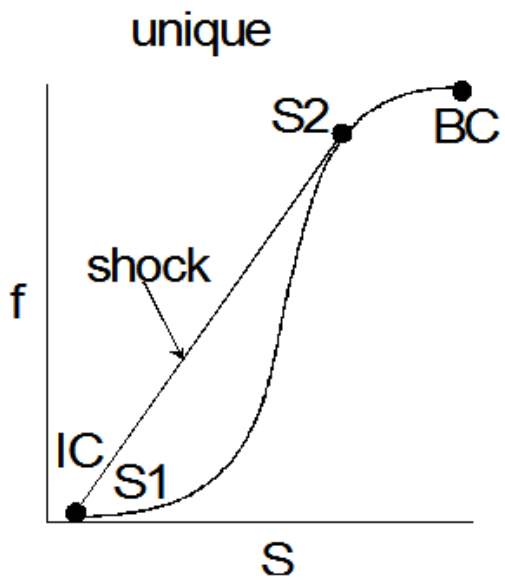
$\Delta f / \Delta S$ is the cord slope of the f versus S curve between S_1 and S_2.

The conservation equation for the shock shows the velocity to be equal to the cord slope between S_1 and S_2 but does not in itself determine S_1 and S_2. To determine S_1 and S_2, we must apply the rule that the waves must have non-decreasing velocity in the direction of flow. The following figure is a solution that is **not admissible**. This solution is not admissible because the velocity of the saturation values (slope) between the IC and $S1$ are less than that of the shock and the velocity of the shock (cord slope) is less than that of the saturation values immediately behind the shock.

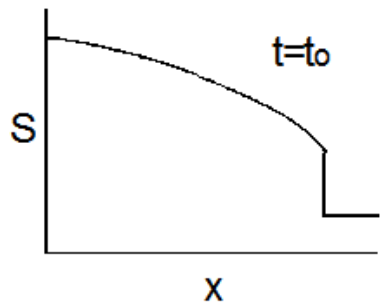


This solution is admissible in that the velocity in nondecreasing in going from the BC to the IC. However, it in **not unique**. Several values of $S2$ will give admissible solutions. Suppose that the value shown here is **a** solution. Also suppose that dispersion across the shock causes the presence of other values of S between $S1$ and $S2$. There are some values of S that will have a velocity (slope) greater than that of the shock shown here. These values of S will overtake $S2$ and the shock will go the these values of S. This will continue until there is no value of S that has a velocity greater than that of the shock to that point. At this point the velocity of the saturation value and that of the shock are equal. On the graphic construction , the cord will be tangent to the curve at this point. This is the **unique** solution in the presence of a small amount of dispersion.

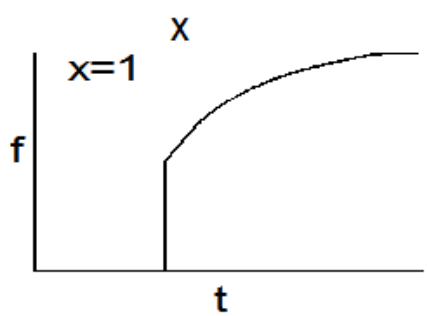




Composition (Saturation) Profile The composition (saturation) profile is the composition distribution existing in the system at a given time.



Composition or Flux History: The composition or flux appearing at a given point in the system, e.g., $x = 1$.



Summary of Equations

The dimensionless velocity that a saturation value propagates is given by the following equation.
Equation:

$$\left(\frac{dx}{dt}\right)_{dS=0} = \frac{df}{dS}(S)$$

With uniform initial and boundary conditions, the origin of all changes in saturation is at $x = 0$ and $t = 0$. If $f(S)$ depends only on S and not on x or t, then the trajectories of constant saturation are straight line determined by integration of the above equation from the origin.
Equation:

$$\begin{aligned} x(S) &= \left(\tfrac{dx}{dt}\right)_{dS=0} t = \tfrac{df}{dS}(S)\ t \\ x(\Delta S) &= \left(\tfrac{dx}{dt}\right)_{\Delta S} t \quad = \tfrac{\Delta f}{\Delta S}\ t \end{aligned}$$

These equations give the trajectory for a given value of S or for the shock. By evaluating these equations for a given value of time these equations give the saturation profile.

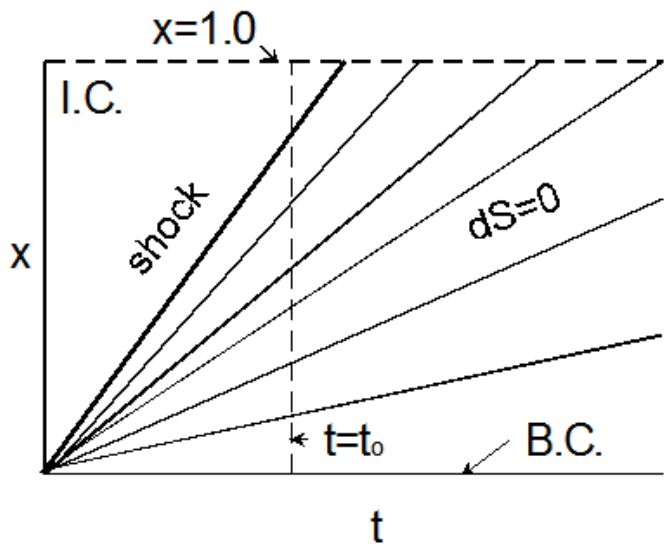
The saturation history can be determined by solving the equations for t with a specified value of x, e.g. $x = 1$.
Equation:

$$t(\Delta S) = \frac{x}{\frac{\Delta f}{\Delta S}}, \qquad x = 1$$
$$t(S) = \frac{x}{\frac{df}{dS}(S)}, \qquad x = 1$$

The **breakthrough time**, t_{BT}, is the time at which the fastest wave reaches $x = 1.0$. The flux history (fractional flow history) can be determined by calculating the fractional flow that corresponds to the saturation history.

Summary of Diagrams

The relationship between the diagrams can be illustrated in a diagram for the trajectories. The profile is a plot of the saturation at $t = t_o$. The history at $x = 1.0$ is the saturation or fractional flow at $x = 1$. In this illustration, the shock wave is the fastest wave. Ahead of the shock is a region of constant state that is the same as the initial conditions.



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Laminar Flows with Dependence on One Dimension

- Couette flow
  - Planar Couette flow
  - Cylindrical Couette flow
  - Planer rotational Couette flow
- Hele-Shaw flow
- Poiseuille flow
  - Friction factor and Reynolds number
  - Non-Newtonian fluids
- Steady film flow down inclined plane
- Unsteady viscous flow
  - Suddenly accelerated plate
  - Developing Couette flow

Reading Assignment
Chapter 2 of BSL, *Transport Phenomena*

One-dimensional (1-D) flow fields are flow fields that vary in only one spatial dimension in Cartesian coordinates. This excludes turbulent flows because it cannot be one-dimensional. Acoustic waves are an example of 1-D compressible flow. We will concern ourselves here with incompressible 1-D flow fields that result from axial or planar symmetry. Cartesian, 1-D incompressible flows do not have a velocity component (other than possibly a uniform translation) in the direction of the spatial dependence because of the condition of zero divergence. Thus the nonlinear convective derivative disappears from the equations of motion in Cartesian coordinates. They may not disappear with curvilinear coordinates.
Equation:

$$\begin{aligned}
&\mathbf{v} = \mathbf{v}(x_3) \\
&\nabla \bullet \mathbf{v} = 0 \quad \Rightarrow \frac{\partial v_3}{\partial x_3} = 0 \\
&v_3\left(x_3 = 0\right) = 0 \quad \Rightarrow v_3 = 0 \\
&\mathbf{v} \bullet \nabla \mathbf{v} = v_i\, v_{j,i} = v_3 \frac{\partial v_j}{\partial x_3} = 0 \\
&\rho \frac{\partial v_j}{\partial t} = -\nabla p + \rho\, \mathbf{f} - \nabla_3 \bullet \tau = -\nabla p + \rho\, \mathbf{f} + \mu \frac{\partial^2 v_j}{\partial x_3^2}, \quad j = 1,\, 2
\end{aligned}$$

We can demonstrate that this relation may not apply in curvilinear coordinates by considering an example with cylindrical polar coordinates. Suppose that the only nonzero component of velocity is in the θ direction and the only spatial dependence is on the r coordinate. The radial component of the convective derivative is non-zero due to centrifugal forces.

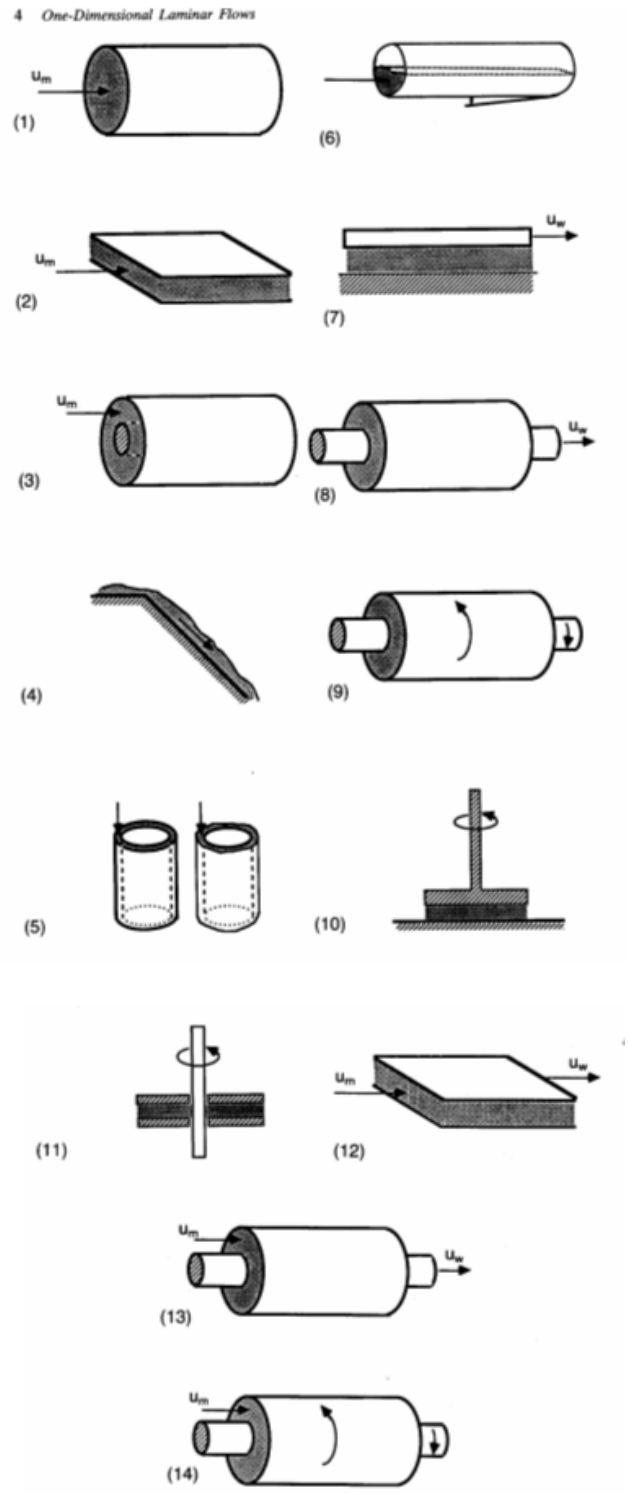
Equation:

$$\mathbf{v} = [0,\ v_\theta(r),\ 0]$$
$$[\mathbf{v} \bullet \nabla \mathbf{v}]_r = -\frac{v_\theta^2}{r}$$

The flows can be classified as either forced flow resulting from the gradient of the pressure or the potential of the body force or induced flow resulting from motion of one of the bounding surfaces.

Some flow fields that result in 1-D flow are listed below and illustrated in the following figure (Churchill, 1988)

1. Forced flow through a round tube
2. Forced flow between parallel plates
3. Forced flow through the annulus between concentric round tubes of different diameters
4. Gravitational flow of a liquid film down an inclined or vertical plane
5. Gravitational flow of a liquid film down the inner or outer surface of a round vertical tube
6. Gravitational flow of a liquid through an inclined half-full round tube
7. Flow induced by the movement of one of a pair of parallel planes
8. Flow induced in a concentric annulus between round tubes by the axial movement of either the outer or the inner tube
9. Flow induced in a concentric annulus between round tubes by the axial rotation of either the outer or the inner tube
10. Flow induced in the cylindrical layer of fluid between a rotating circular disk and a parallel plane
11. Flow induced by the rotation of a central circular cylinder whose axis is perpendicular to parallel circular disks enclosing a thin cylindrical layer of fluid
12. Combined forced and induced flow between parallel plates
13. Combined forced and induced longitudinal flow in the annulus between concentric round tubes
14. Combined forced and rotationally induced flow in the annulus between concentric round tubes



Geometry and conditions that produce
one-dimensional velocity fields
(Churchill, 1988)

Couette Flow

The flows when the fluid between two parallel surfaces are induced to flow by the motion of one surface relative to the other is called *Couette flow*. This is the generic shear flow that is used to illustrate Newton's law of viscosity. Pressure and body forces balance each other and at steady state the equation of motion simplify to the divergence of the viscous stress tensor or the Laplacian of velocity in the case of a Newtonian fluid.

Planar Couette flow. (case 7).
Equation:

$$\frac{d\tau}{dx_3} = -\mu \frac{d^2 v_j}{dx_3^2} = 0, \quad j = 1,\, 2$$

The coordinates system can be defined so that $\mathbf{v} = 0$ at $x_3 = 0$ and the j component of velocity is non-zero at $x_3 = L$.
Equation:

$$\begin{aligned} v_j &= 0, \quad x_3 = 0 \\ v_j &= U_j, \quad x_3 = L, \quad j = 1,\, 2 \end{aligned}$$

The velocity field is
Equation:

$$v_j = \frac{U_j\, x_3}{L}$$

The shear stress can be determined from Newton's law of viscosity.
Equation:

$$\tau_{j3} = -\mu \frac{dv_j}{dx_3} = -\mu \frac{U_j}{L}, \quad j = 1,\, 2$$

<u>Cylindrical Couette flow.</u> The above example was the translational movement of two planes relative to each other. Couette flow is also possible in the annular gap between two concentric cylindrical surfaces (cases 8 and 9) if secondary flows do not occur due to centrifugal forces. We use cylindrical polar coordinates rather than Cartesian and <u>assume</u> vanishing Reynolds number. The only independent variable is the radius.
Equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) = 0$$
$$[\nabla \bullet \tau]_r = \frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rr}) - \frac{\tau_{\theta\theta}}{r}, \quad \text{may not vanish if Reynolds number is high}$$
$$[\nabla \bullet \tau]_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\theta}\right) + \frac{\tau_{\theta r}-\tau_{r\theta}}{r} = 0, \quad \text{may not vanish if Reynolds number is high}$$
$$[\nabla \bullet \tau]_z = \frac{1}{r}\frac{\partial}{\partial r}(r\,\tau_{rz}) = 0$$
$$\left.\begin{aligned} \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\,v_\theta)\right] &= 0 \\ \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) &= 0 \end{aligned}\right\} \quad \text{Newtonian fluid}$$

The stress profile can be calculated by integration.
Equation:

$$r^2\tau_{r\theta} = r^2\tau_{r\theta}\big|_{r=r_1} = r^2\tau_{r\theta}\big|_{r=r_2}$$
$$r\tau_{rz} = r\tau_{rz}\big|_{r=r_1} = r\tau_{rz}\big|_{r=r_2}$$

The boundary conditions on velocity depend on whether the cylindrical surfaces move relative to each other as a result of rotation, axial translation, or both.
Equation:

$$\mathbf{v} = 0, \quad r = r_1$$
$$\mathbf{v} = (0, U_\theta, U_z), \quad r = r_2$$

The velocity field for cylindrical Couette flow of a Newtonian fluid is .
Equation:

$$v_\theta = \frac{U_\theta}{\frac{r_2}{r_1}-\frac{r_1}{r_2}}\left(\frac{r}{r_1} - \frac{r_1}{r}\right)$$
$$v_z = \frac{U_z}{\log(r_2/r_1)}\ \log\,(r/r_1)$$

<u>Planer rotational Couette flow.</u> The parallel plate viscometer has the configuration shown in case 10. The system is not strictly 1-D because the velocity of one of the surfaces is a function of radius. Also, there is a centrifugal force present near the rotating surface but is absent at the stationary surface. However, if the Reynolds number is small enough that secondary flows do not occur, then the velocity at a given value of the radius may be approximated as a function of only the z distance in the gap. The differential equations at zero Reynolds number are as follows.
Equation:

$$[\nabla \bullet \tau]_\theta = \frac{\partial \tau_{\theta z}}{\partial z} = 0$$
$$\frac{\partial^2 v_\theta}{\partial z^2} = 0, \quad \text{Newtonian fluid}$$

Suppose the bottom surface is stationary and the top surface is rotating. Then the boundary conditions are as follows.
Equation:

$$\begin{aligned} v_\theta &= 0, \qquad z = 0 \\ v_\theta &= 2\pi\, r\, \Omega, \quad z = L \end{aligned}$$

The stress and velocity profiles are as follows.
Equation:

$$\begin{aligned} &\tau_{\theta z} = \tau_{\theta z}(r) = -2\pi\, r\, \Omega\, \mu(r)/L \\ &v_\theta = 2\pi\, r\, \Omega\, z/L, \quad \text{Newtonian fluid} \end{aligned}$$

The stress is a function of the radius and if the fluid is non-Newtonian, the viscosity may be changing with radial position.

Plane-Poiseuille and Hele-Shaw flow

Forced flow between two stationary, parallel plates, case 2, is called plane-Poiseuille flow or if the flow depends on two spatial variables in the plane, it is called Hele-Shaw flow. The flow is forced by a specified flow rate or a specified pressure or gravity potential gradient. The pressure and gravitational potential can be combined into a single variable, P.
Equation:

$$\begin{aligned} &-\nabla p + \rho\, \mathbf{f} = -\nabla P \\ &\text{where} \\ &P = p + \rho\, g\, h \end{aligned}$$

The product gh is the gravitational potential, where g is the acceleration of gravity and h is distance upward relative to some datum. The pressure, p, is also relative to a datum, which may be the datum for h.

The primary spatial dependence is in the direction normal to the plane of the plates. If there is no dependence on one spatial direction, then the flow is truly one-dimensional. However, if the velocity and pressure gradients have components in two directions in the plane of the plates, the flow is not strictly 1-D and nonlinear, inertial terms will be present in the equations of motion. The significance of these terms is quantified by the Reynolds number. If the flow is steady, and the Reynolds number negligible, the equations of motion are as follows.
Equation:

$$
\begin{aligned}
0 &= -\frac{\partial P}{\partial x_j} - \frac{\partial \tau_{j3}}{\partial x_3}, \qquad j = 1,\, 2 \\
0 &= -\frac{\partial P}{\partial x_3} - 0 \\
0 &= -\frac{\partial P}{\partial x_j} + \mu \frac{\partial^2 v_j}{\partial x_3^2}, \quad j = 1,\, 2, \quad \text{Newtonian fluid}
\end{aligned}
$$

Let h be the spacing between the plates and the velocity is zero at each surface.
Equation:

$$
v_j = 0, \quad x_3 = 0,\, h \quad j = 1,\, 2
$$

The velocity profile for a Newtonian fluid in plane-Poiseuille flow is
Equation:

$$
v_j = \frac{h^2}{2\mu} \frac{\partial P}{\partial x_j} \left[\left(\frac{x_3}{h} \right)^2 - \frac{x_3}{h} \right], \quad j = 1,\, 2, \quad 0 \le x_3 \le h
$$

The average velocity over the thickness of the plate can be determined by integrating the profile.
Equation:

$$
\overline{v}_j = -\frac{h^2}{12\mu} \frac{\partial P}{\partial x_j}, \quad j = 1,\, 2
$$

This equation for the average velocity can be written as a vector equation if it is recognized that the vectors have components only in the $(1, 2)$ directions.
Equation:

$$
\overline{\mathbf{v}} = -\frac{h^2}{12\mu} \nabla P, \quad \overline{\mathbf{v}} = \overline{\mathbf{v}}\,(x_1,\, x_2), \qquad \nabla P = \nabla P\,(x_1,\, x_2)
$$

If the flow is incompressible, the divergence of velocity is zero and the potential, P, is a solution of the Laplace equation except where sources are present. If the strength of the sources or the flux at boundaries are known, the potential, P, can be determined from the methods for the solution of the Laplace equation.

We now have the result that the average velocity vector is proportional to a potential gradient. Thus the average velocity field in a Hele-Shaw flow is irrotational. If the fluid is incompressible, the average velocity field is also solenoidal can can be expressed as the curl of a vector potential or the stream function. The average velocity field of Hele-Shaw flow is an physical analog for the irrotational, solenoidal, 2-D flow described by the complex potential. It is also a physical analog for 2-D flow of incompressible fluids through porous

media by Darcy's law and was used for that purpose before numerical reservoir simulators were developed.

Poiseuille Flow

Poiseuille law describes laminar flow of a Newtonian fluid in a round tube (case 1). We will derive Poiseuille law for a Newtonian fluid and leave the flow of a power-law fluid as an assignment. The equation of motion for the steady, developed (from end effects) flow of a fluid in a round tube of uniform radius is as follows.
Equation:

$$
\begin{aligned}
0 &= -\frac{\partial P}{\partial r} \\
0 &= -\frac{\partial P}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}), \quad 0 < r < R \\
0 &= -\frac{\partial P}{\partial z} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right), \quad \text{Newtonian fluid}
\end{aligned}
$$

The boundary conditions are symmetry at $r = 0$ and no slip at $r = R$.
Equation:

$$
\begin{aligned}
&\tau_{rz}\big|_{r=0} = -\mu\frac{\partial v_z}{\partial r}\Big|_{r=0} = 0 \\
&v_z = 0, \quad r = R
\end{aligned}
$$

From the radial component of the equations of motion, P does not depend on radial position. Since the flow is steady and fully developed, the gradient of P is a constant. The z component of the equations of motion can be integrated once to derive the *stress profile* and *wall shear stress*.
Equation:

$$
\begin{aligned}
\tau_{rz} &= \left(-\frac{\partial P}{\partial z}\right)\frac{r}{2}, \quad 0 < r < R \\
\tau_w &= \left(-\frac{\partial P}{\partial z}\right)\frac{R}{2}
\end{aligned}
$$

If the fluid is Newtonian, the equation of motion can be integrated once more to obtain the *velocity profile* and *maximum velocity*.
Equation:

$$
\left.\begin{aligned}
v_z &= \frac{R^2}{4\mu}\left(-\frac{\partial P}{\partial z}\right)\left[1 - \left(\frac{r}{R}\right)^2\right], \quad 0 < r < R \\
v_{z,\,\max} &= \frac{R^2}{4\mu}\left(-\frac{\partial P}{\partial z}\right)
\end{aligned}\right\}, \quad \text{Newtonian fluid}
$$

The *volumetric rate* of flow through the pipe can be determined by integration of the velocity profile across the cross-section of the pipe, i.e., $0 < r < R$ and $0 < \theta < 2\pi$.
Equation:

$$Q = \frac{\pi R^4}{8\mu}\left(-\frac{\partial P}{\partial z}\right), \quad \text{Newtonian fluid}$$

This relation is the *Hagen-Poiseuille law*. If the flow rate is specified, then the potential gradient can be expressed as a function of the flow rate and substituted into the above expressions.

The *average velocity* or volumetric flux can be determined by dividing the volumetric rate by the cross-sectional area.
Equation:

$$\langle v_z \rangle = \frac{R^2}{8\mu}\left(-\frac{\partial P}{\partial z}\right), \quad \text{Newtonian fluid}$$

Before one begins to believe that the Hagen-Poiseuille law is a "law" that applies under all conditions, the following is a list of assumptions are implicit in this relation (BSL, 1960).

a. The flow is laminar-NRe less than about 2100.
b. The density is constant ("incompressible flow").
c. The flow is independent of time ("steady state").
d. The fluid is Newtonian.
e. End effects are neglected-actually an "entrance length" (beyond the tube entrance) on the order of Le = 0.035D NRe is required for build-up to the parabolic profiles; if the section of pipe of interest includes the entrance region, a correction must be applied. The fractional correction introduced in either P or Q never exceeds Le/L if L > Le.
f. The fluid behaves as a continuum-this assumption is valid except for very dilute gases or very narrow capillary tubes, in which the molecular mean free path is comparable to the tube diameter ("slip flow" regime) or much greater than the tube diameter ("Knudsen flow" or "free molecule flow" regime).
g. There is no slip at the wall-this is an excellent assumption for pure fluids under the conditions assumed in (f).

<u>Friction factor and Reynolds number.</u> Because pressure drop in pipes is commonly used in process design, correlation expressed as friction factor versus Reynolds number are available for laminar and turbulent flow. The Hagen-Poiseuille law describes the laminar flow portion of the correlation. The correlations in the literature differ when they use different definitions for the friction factor. Correlations are usually are usually expressed in terms of the Fanning friction factor and the Darcy-Weisbach friction factor.
Equation:

$$f_{SP} \equiv \frac{\tau_w}{\rho\, u_m^2}, \quad \text{Stanton} - \text{Pannell friction factor}$$
$$f_F \equiv \frac{2\tau_w}{\rho\, u_m^2}, \quad \text{Fanning friction factor}$$
$$f_{DW} \equiv \frac{8\tau_w}{\rho\, u_m^2}, \quad \text{Darcy} - \text{Weisbach friction factor (Moody)}$$
$$u_m = \langle v \rangle, \quad \text{mean velocity}$$

The Reynolds number is expressed as a ratio of inertial stress and shear stress.
Equation:

$$N_{\mathrm{Re}} = \frac{\rho\, u_m^2}{\mu\, u_m/D} = \frac{\rho\, u_m\, D}{\mu} = \frac{2\rho\, u_m\, R}{\mu}$$

Both the friction factor and the Reynolds number have as a common factor, the kinetic energy per unit volume ρu_m^2 . This factor may be eliminated between the two equations to express the friction factor as a function of the Reynolds number.
Equation:

$$f_{SP} = \frac{1}{N_{RE}} \frac{\tau_w}{\mu\, u_m/D}$$
$$f_F = \frac{2}{N_{RE}} \frac{\tau_w}{\mu\, u_m/D}$$
$$f_{DW} = \frac{8}{N_{RE}} \frac{\tau_w}{\mu\, u_m/D}$$

Recall the expressions derived earlier for the wall shear stress and the average velocity for a Newtonian fluid and substitute into the above expressions.
Equation:

$$f_{SP} = \frac{8}{N_{\mathrm{Re}}}$$
$$f_F = \frac{16}{N_{\mathrm{Re}}}$$
$$f_{DW} = \frac{64}{N_{\mathrm{Re}}}$$

Correlation of friction factor versus Reynolds number appear in the literature with all three definitions of the friction factor and usually without a subscript to denote which definition is being used.

<u>Non-Newtonian fluids.</u> The velocity profiles above were derived for a Newtonian fluid. A constitutive relation is necessary to determine the velocity profile and mean velocity for non-Newtonian fluids. We will consider the cases of a Bingham model fluid and a power-law or Ostwald-de Waele model fluid. The constitutive relations for these fluids are as follows.
Equation:

$$\text{The Bingham Model}$$
$$\tau_{yx} = -\mu_o \frac{dv_x}{dy} \pm \tau_o, \quad \text{if } |\tau_{yx}| > \tau_o$$
$$\frac{dv_x}{dy} = 0, \qquad \text{if } |\tau_{yx}| < \tau_o$$
$$\text{The Ostwald} - \text{de Waele (power} - \text{law) Model}$$
$$\tau_{yx} = -m \left| \frac{dv_x}{dy} \right|^{n-1} \frac{dv_x}{dy}$$

The power-law model is an empirical model that is often valid over an intermediate range of shear rates. At very low and very high shear rates limiting values of viscosity are approached.

Assignment 8.1
Flow in annular space between concentric cylinders as function of relative translation, rotation, potential gradient, flow or no-net flow. Assume incompressible, Newtonian fluid with small Reynolds number. The outer radius has zero velocity. Parameters:

R_2	outer radius
R_1	inner radius, maybe zero
$\frac{\partial P}{\partial z}$	potential gradient, may be zero
V_{z1}	translation velocity of inner radius, may be zero
$V_{\theta 1}$	rotational velocity of inner radius
q	net flow rate, may be zero

a. Express dimensionless velocity as a function of the dimensionless radius and dimensionless groups. Plot the following cases:

Table of cases to plot					
Case	R_1/R_2	$\partial P/\partial z$	v_{z1}	$v_{\theta 1}$	q

1	0	$\neq 0$			$\neq 0$
2	0.5	$\neq 0$	0	0	$\neq 0$
3	0.5	$= 0$	$\neq 0$	0	$\neq 0$
4	0.5	$\neq 0$	$\neq 0$	0	0
5	0.5	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$

b. What is the net flow if the inner cylinder is translating and the pressure gradient is zero?
c. What is the pressure gradient if the net flow is zero? Plot the velocity profile for this case.

Assignment 8.2
Capillary flow of power-law model fluid. Calculate the following for a power-law model fluid (see hint in BSL, 1960).

a. Calculate and plot the velocity profile, normalized with respect to the mean velocity for $n = 1, 0.67, 0.5, \text{and} 0.33$.
b. Derive an expression corresponding to Poiseuille law.
c. Derive the same relation between friction factor and Reynolds number as for Newtonian flow by defining a modified Reynolds number for power-law fluids.

Steady film flow down inclined plane

Steady film flow down an inclined plane corresponds to case 4 (Churchill, 1988) or Section 2.2; Flow of a Falling Film (BSL, 1960). These flows occur in chemical processing with falling film sulfonation reactors, evaporation and gas adsorption, and film-condensation heat transfer. It is assumed that the flow is steady and there is no dependence on distance in the plane of the surface due to entrance effects, side walls, or ripples. The Reynolds number must be small enough for ripples to be avoided. The configuration will be similar to that of BSL except $x = 0$ corresponds to the wall and the thickness is denoted by h rather than δ.

It is assumed that the gas has negligible density compared to the liquid such that the pressure at the gas-liquid interface can be assumed to be constant. The potential gradient in the plane of the film is constant and can be expressed either in term of the angle from the vertical, β , or the angle from the horizontal, α.
Equation:

$$-\nabla P = \rho g \ \cos \ \beta = \rho g \ \sin \ \alpha, \quad \alpha = \pi/2 - \beta$$

The equations of motion are as follows.
Equation:

$$
\begin{aligned}
0 &= \frac{dP}{dx} \\
0 &= \rho\, g\ \cos\ \beta - \frac{d\tau_{xz}}{dx} \\
0 &= \rho\, g\ \cos\ \beta + \mu \frac{d^2 v_z}{dx^2}, \quad \text{Newtonian fluid}
\end{aligned}
$$

The boundary conditions are zero stress at the gas-liquid interface and no slip at the wall.
Equation:

$$
\begin{aligned}
v_z &= 0, \quad x = 0 \\
\tau_{xz} &= 0, \quad x = h
\end{aligned}
$$

The shear stress profile can be determined by integration and application of the zero stress boundary condition.
Equation:

$$
\begin{aligned}
\tau_{xz} &= \rho\, g\, h\ \cos\ \beta \left(\frac{x}{h} - 1\right), \quad 0 \le x \le h \\
\tau_w &= -\rho\, g\, h\ \cos\ \beta
\end{aligned}
$$

The velocity profile for a Newtonian fluid can be determined by a second integration and application of the no slip boundary condition.
Equation:

$$
\begin{aligned}
v_z &= \frac{\rho\, g\, h^2 \cos\beta}{2\mu} \left[\frac{2x}{h} - \left(\frac{x}{h}\right)^2\right], \quad 0 \le x \le h \\
v_{z,\,\max} &= \frac{\rho\, g\, h^2 \cos\beta}{2\mu}
\end{aligned}
$$

The average velocity and volumetric flow rate can be determined by integration of the velocity profile over the film thickness.
Equation:

$$
\begin{aligned}
\langle v_z \rangle &= \frac{\rho\, g\, h^2 \cos\beta}{3\mu} \\
Q &= \frac{\rho\, g\, W\, h^3 \cos\beta}{3\mu}
\end{aligned}
$$

The film thickness, h, can be given in terms of the average velocity, the volume rate of flow, or the mass rate of flow per unit width of wall $(\Gamma = \rho\, h\, \langle v_z \rangle)$:
Equation:

$$
h = \sqrt{\frac{3\mu \langle v_z \rangle}{\rho\, g\ \cos\ \beta}} = \sqrt[3]{\frac{3\mu\, Q}{\rho\, g\, W\ \cos\ \beta}} = \sqrt[3]{\frac{3\mu\, \Gamma}{\rho^2\, g\ \cos\ \beta}}
$$

Unsteady viscous flow

Suddenly accelerated plate. (BSL, 1960) A semi-infinite body of liquid with constant density and viscosity is bounded on one side by a flat surface (the xz plane). Initially the fluid and solid surface is at rest; but at time $t = 0$ the solid surface is set in motion in the positive x-direction with a velocity U. It is desired to know the velocity as a function of y and t. The pressure is hydrostatic and the flow is assumed to be laminar.

The only nonzero component of velocity is $v_x = v_x\,(y, t)$. Thus the only non-zero equation of motion is as follows.
Equation:

$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}, \quad y > 0,\ t > 0$$

The initial condition and boundary conditions are as follows.
Equation:

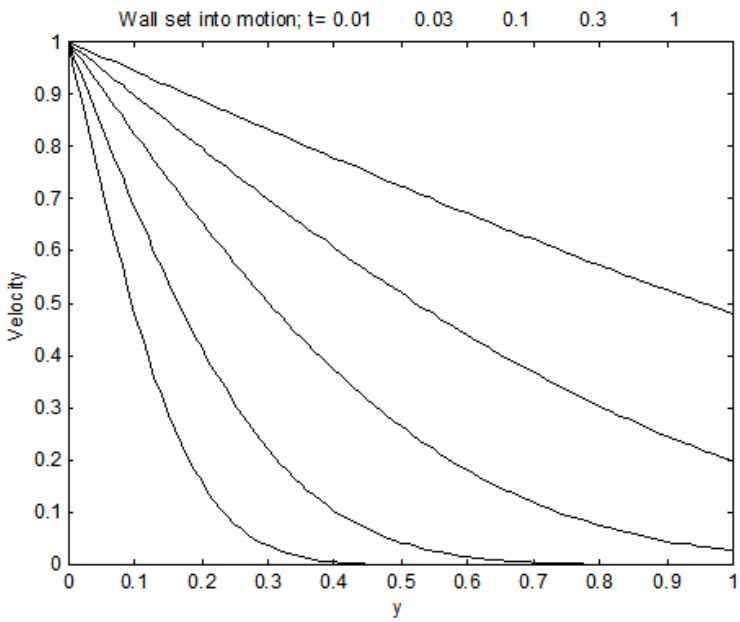
$$\begin{aligned} v_x &= 0, \quad t = 0,\ y > 0 \\ v_x &= U, \quad y = 0,\ t > 0 \\ v_x &= 0, \quad y \to \infty,\ t > 0 \end{aligned}$$

If we normalize the velocity with respect to the boundary condition, we see that this is the same parabolic PDE and boundary condition as we solved with a similarity transformation. Thus the solution is
Equation:

$$v_x = U\, erfc\left(\frac{y}{\sqrt{4\mu\, t/\rho}} \right)$$

The presence of the ratio of viscosity and density, the kinematic viscosity, in the expression for the velocity implies that both viscous and inertial forces are operative.

The velocity profiles for the wall at $y = 0$ suddenly set in motion is illustrated below.



<u>Developing Couette flow.</u> The transient development to the steady-state Couette flow discussed earlier can now be easily derived. We will let the plane $y = 0$ be the surface with zero velocity and let the velocity be specified at $y = L$. The initial and boundary conditions are as follows.

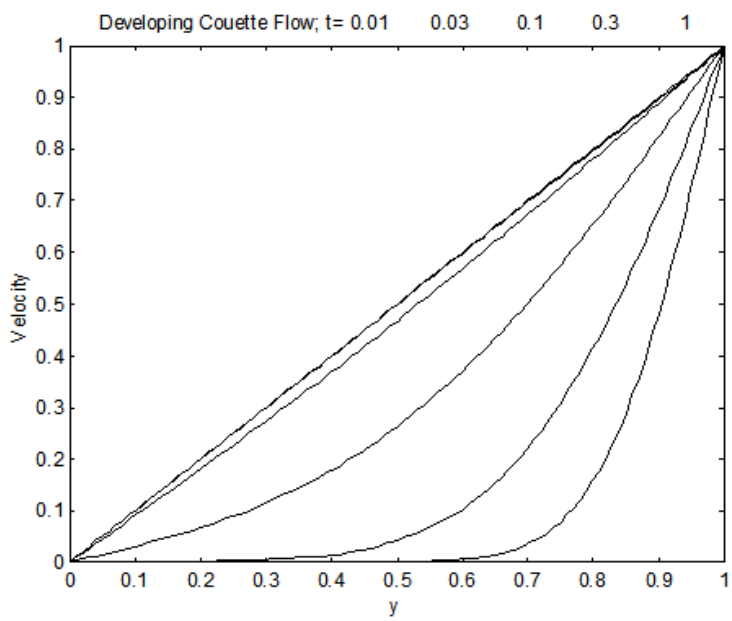
Equation:

$$
\begin{aligned}
&v_x = 0, \quad t = 0, \quad 0 < y < L \\
&v_x = 0, \quad y = 0, \quad t > 0 \\
&v_x = U, \quad y = L, \quad t > 0 \\
&\text{or} \\
&v_x = 0, \quad t = 0, \quad -L < y < L \\
&v_x = -U, \quad y = -L, \quad t > 0 \\
&v_x = U, \quad y = L, \quad t > 0
\end{aligned}
$$

It should be apparent that the two formulations of the boundary conditions give the same solution. However, the latter gives a clue how one should obtain a solution. The solution is antisymmetric about $y = 0$ and the zero velocity condition is satisfied. A series of additional terms are needed to satisfy the boundary conditions at $y = \pm L$. The solution is

Equation:

$$
\begin{aligned}
&v_x = U \sum_{n=0}^{\infty} \left\{ erfc\left[\frac{(2n+1)L-y}{\sqrt{4\nu t}}\right] - erfc\left[\frac{(2n+1)L+y}{\sqrt{4\nu t}}\right] \right\} \\
&\nu = \frac{\mu}{\rho}
\end{aligned}
$$



Asignment. 8.3

Flow of a fluid with a suddenly applied constant wall stress. This problem is similar to that of flow $(-\infty < x < \infty,\ y > 0,\ t > 0)$ near a wall $(-\infty < x < \infty,\ y = 0)$ suddenly set into motion, except that the shear stress at the wall is constant rather than the velocity. Let the fluid be at rest before $t = 0$. At time $t = 0$ a constant force is applied to the fluid at the wall, so that the shear stress τ_{yx} takes on a new constant value τ_o at $y = 0$ for $t > 0$.

a. Start with the continuity and Navier-Stokes equations and eliminate the terms that are identically zero. Differentiate the resulting equation with respect to distance from the wall and multiply by viscosity to derive an equation for the evolution of the shear stress. List all assumptions.
b. Write the boundary and initial conditions for this equation.
c. Solve for the time and distance dependence of the shear stress. Sketch the solution.
d. Calculate the velocity profile from the solution of the shear stress. The following equation will be helpful.
e. Suppose the fluid is not infinite but rather there is another wall at a distance 2h away from the original wall and it was also set into motion but in the opposite direction with the same wall shear stress. What are the stress and velocity profiles for the fluid between the two walls? Sketch and express as series solutions.
f. What are the steady state stress and velocity profiles for the problem of part (e)? Sketch and express as analytical solutions.

Multidimensional Laminar Flow

Reading assignment
Chapter 4 in BSL, Transport Phenomena

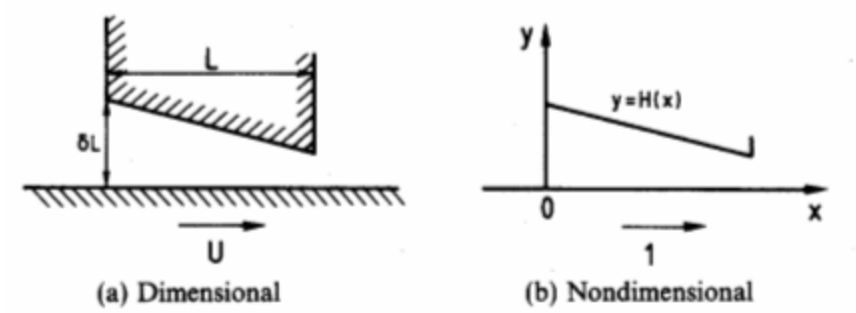
Lubrication and Film Flow

We already had two examples of flow in gaps that could be a thin film; Couette flow, and the steady, draining film. Here we will see that when the dimension of the gap or film thickness is small compared to other dimensions of the system, the Navier-Stokes equations simplify relatively and simple, classical solutions are possible. In Chapter 6, we saw that when the dimension of the gap or film in the x_3 direction is small and the Reynolds number is small, the equations of motion reduce to the following.
Equation:

$$0 = -\frac{\partial P}{\partial x_3} + O\left(\frac{h_o}{L}\right)^2 + O\left(\mathrm{Re}\right)$$
$$0 = -\nabla_{12} P + \mu \frac{\partial^2 v_{12}}{\partial x_3^2} + O\left(\frac{h_o}{L}\right)^2 + O\left(\mathrm{Re}\right)$$

In the above equations, the subscript, 12, denote components in the plane of the gap or film. When the thickness is small enough, one wall of the gap or film can be treated as a plane even if it is curved with a radius of curvature that is large compared to the thickness.

<u>Lubrication flow with slider bearings.</u> (Ockendon and Ockendon, 1995) Bearings function preventing contact between two moving surfaces by the flow of the lubrication fluid between the surfaces. The generic example of lubrication flow is illustrated with the slider bearing.



A two-dimensional bearing is shown in which the plane of $y = 0$ moves with constant velocity U in the x-direction and the top of the bearing (the slider) is fixed. The variables are nondimensionalised with respect to U, the length L of the bearing, and a characteristic gap-width, h_o, so that the position of the slider is given in the dimensionless variables. Again, referring back to Chapter 6, the dimensionless variables for this problem may be the following.

Equation:

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{h_o}, \quad h(x) = \frac{H(x)}{h_o}$$
$$u^* = \frac{u}{U}, \quad v^* = \frac{v}{U}\frac{L}{h_o}$$
$$P^* = \frac{h_o^2 p}{\mu L U}$$

Henceforth, the variables will be dimensionless with the * dropped. The boundary conditions are as follows.

Equation:

$$u = 1, \quad v = 0, \quad y = 0$$
$$u = 0, \quad v = 0, \quad y = h(x)$$
$$P = 0, \qquad x = 0,\ 1$$

The pressure can not suddenly equal the ambient pressure as assumed here because entrance and exit effects, but these will be neglected here. In reality, there may be a high pressure at the entrance of the bearing as a result of the liquid being scraped from the surface. The low pressure at the

exit of the bearing may result in gas flowing in to equalize the pressure or cavitation may occur.

The dimensionless equations of motion and continuity equation are now as follows.

Equation:

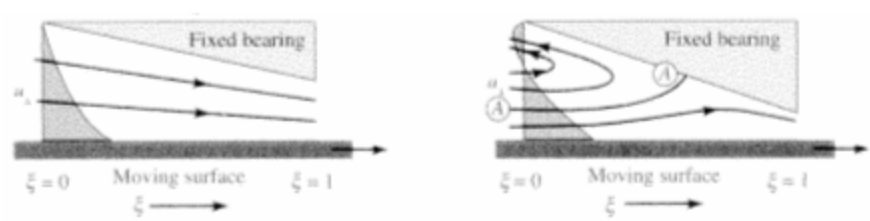
$$
\begin{aligned}
0 &= -\frac{\partial P}{\partial y}, \quad \Rightarrow P = P(x) \\
0 &= -\frac{dP}{dx} + \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial u}{\partial x} &+ \frac{\partial v}{\partial y} = 0
\end{aligned}
$$

Integration of the equation of motion over the gap-thickness gives the velocity profile for a particular value of x.

Equation:

$$
u = \frac{h^2}{2}\frac{dP}{dx}\left[\left(\frac{y}{h}\right)^2 - \frac{y}{h}\right] + 1 - \frac{y}{h}
$$

Notice that this profile is a combination of a profile due to forced flow (pressure gradient) and that due to induced flow (movement of wall). The velocity may pass through zero somewhere in the profile if the two contributions are in opposite directions. This is illustrated in the following figure.



Velocity profile in slider bearing (Middleman, 1998)

Integration of the continuity equation over the thickness gives,
Equation:

$$\begin{aligned} 0 &= \int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dy \\ &= \int_0^h \frac{\partial u}{\partial x}\, dy + v\big|_0^h \\ &= \int_0^h \frac{\partial u}{\partial x}\, dy \end{aligned}$$

The latter integral can be expressed as follows.
Equation:

$$\begin{aligned} \int_0^h \frac{\partial u}{\partial x}\, dy &= \frac{d}{dx} \int_0^{h(x)} u\, dy - \frac{dh}{dx} u \Big|^{y=h} \\ &= \frac{d}{dx} \int_0^{h(x)} u\, dy \\ &= 0 \end{aligned}$$

Substituting the velocity profile into the above integral gives us the Reynolds equation for lubrication flow.
Equation:

$$\frac{d}{dx} \left(\frac{h^3}{6} \frac{dP}{dx} \right) - \frac{dh}{dx} = 0$$

Integration of this equation gives,
Equation:

$$\frac{h^3}{6} \frac{dP}{dx} = h + C$$

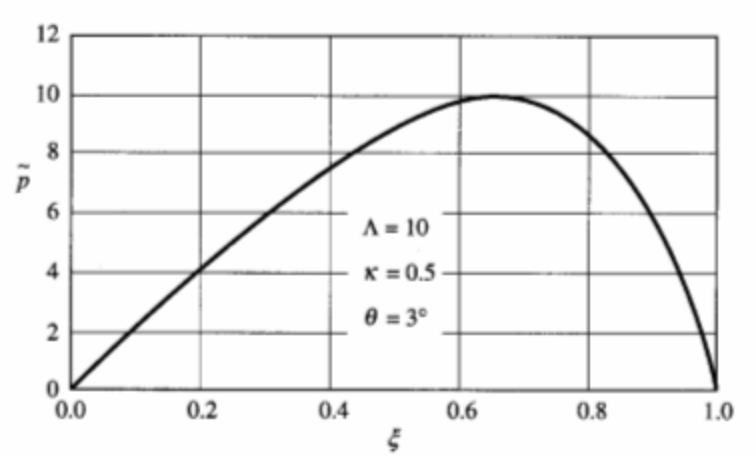
A second integration gives,
Equation:

$$P(x) = 6\int_0^x \frac{h(x') + C}{h^3(x')} dx'$$

where the constant of integration has to satisfy the boundary conditions,
Equation:

$$\int_0^1 \frac{h(x')+C}{h^3(x')} dx' = 0$$

$$C = \frac{-\int_0^1 \frac{dx'}{h^2(x')}}{\int_0^1 \frac{dx'}{h^3(x')}}$$

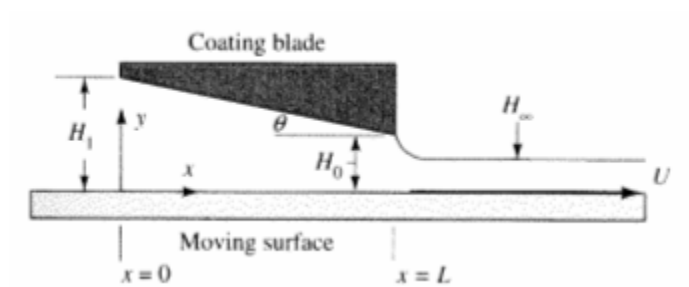


Pressure profile is slider bearing (Middleman, 1998).
$\kappa = H_L/H_o, \quad \Lambda = L/H_o, \quad \tan\theta = (H_o - H_L)/L$

The role of the lubrication layer is to maintain a separation of the two surfaces in the presence of a load such that asperities (roughness) on the surfaces do not make contact. The load on the bearing is equal to the integral of the normal stress over the bearing surface. If the change in gap-thickness is small compared to the length, as assumed here, the normal stress is approximately equal to the pressure. If the gap-thickness is

monotone decreasing, the pressure will be greater than ambient pressure inside the bearing. However, if the gap-thickness in monotone increasing, the pressure will be less than ambient in the bearing and it will have no load bearing capacity. If the gap-thickness is not monotone, then the pressure may be greater than ambient in some places and less than ambient in other places. If the bearing is designed to be load bearing, then a long section of decreasing gap-thickness and a short section of increasing thickness is desired. If the bearing is designed to be a scraper as piston rings then both sections of changing thickness will be short as to limit the amount of liquid passing through the gap. Gas entering the low-pressure region at the exit of the bearing surface prevents bearing surface contact from negative pressures.

The analysis for the slider bearing can also be used to design an apparatus for depositing a uniform coating of a liquid on a substrate.

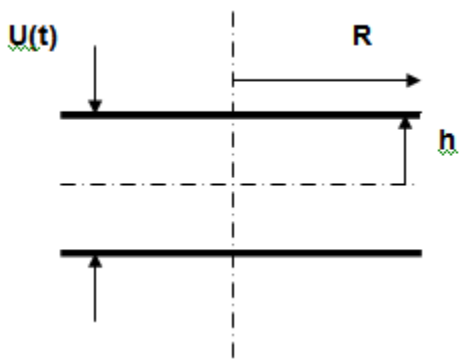


Coating flow (Middleman, 1998)

<u>Squeeze films.</u> When two objects approach each other in a fluid their relative velocity is slowed as the resistance increases for the fluid leaving the gap. Here the relative velocity of the surface is in the normal direction rather than in the parallel direction as in the case of the slider bearing. This type of flow is important in the coalescence of emulsion droplets or foam bubbles. In the case of coalescence, hydrodynamics govern the dynamics of

the approach of the surfaces to each other until the thinning is accelerated or retarded by surface forces (i.e., disjoining pressure).

We will derive the classical Reynolds drainage of a liquid between two parallel disks of radius R approaching each other. The configuration of the system is shown below.



Schematic of film drainage between parallel disks

The system is symmetrical about its axis and the mid-plane. The thickness, h, is one-half of the distance between the disks and the velocity of each disk, U, is one-half of the approach velocity of the two disks. This nomenclature may be awkward but with this nomenclature, the solution also applies to the thinning of a liquid film between a solid surface and a gas bubble having zero shear stress at the interface. The velocity of approach of the disks may not be constant but rather the force pressing the disks together may be constant. Because of the symmetry, we will analyze the upper half-space with cylindrical polar coordinates. The system is axisymmetrical so the independent variables are r, z, t. The equations of motion and continuity equation in cylindrical polar coordinates are

Equation:

$$0 = -\frac{\partial P}{\partial z} + O\left(\frac{h}{R}\right)^2 + O\left(\mathrm{Re}\right)$$
$$0 = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2} + O\left(\frac{h}{R}\right)^2 + O\left(\mathrm{Re}\right)$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\, v_r\right) + \frac{\partial v_z}{\partial z} = 0$$

The boundary conditions are
Equation:

$$\frac{\partial P}{\partial r} = 0, \quad r = 0$$
$$P = 0, \quad r = R$$
$$v_z = \frac{\partial v_r}{\partial z} = 0, \quad z = 0$$
$$v_r = 0, \quad v_z = -U\left(t\right), \quad z = h\left(t\right)$$

The partial differential equations do not have an explicit dependence on time as time only enters thorough the boundary conditions. Thus the variables will be made dimensionless with respect to the time dependent boundary conditions for the purpose of solving the PDE.
Equation:

$$r^* = \frac{r}{R}, \quad z^* = \frac{z}{h}$$
$$v_r{}^* = \frac{v_r}{U}\frac{h}{R}, \quad v_z{}^* = \frac{v_z}{U}$$
$$P^* = \frac{h^3 P}{\mu R^2 U}$$

The dimensionless equations and boundary conditions with the * dropped are now
Equation:

$$
\begin{aligned}
&0 = -\tfrac{\partial P}{\partial z}, \quad \Rightarrow P = P\,(r) \\
&0 = -\tfrac{dP}{dr} + \tfrac{\partial^2 v_r}{\partial z^2} \\
&\tfrac{1}{r}\,\tfrac{\partial}{\partial r}(r\,v_r) + \tfrac{\partial v_z}{\partial z} = 0 \\
&\tfrac{\partial P}{\partial r} = 0, \quad r = 0 \\
&P = 0, \quad r = 1 \\
&v_z = \tfrac{\partial v_r}{\partial z} = 0, \quad z = 0 \\
&v_r = 0, \quad v_z = -1, \quad z = 1
\end{aligned}
$$

Integration of v_r with respect to z in the equation of motion and applying the boundary conditions results in the velocity profile across the film thickness.

Equation:

$$
v_r = \frac{1}{2}\,\frac{dP}{dr}\left(z^2 - 1\right)
$$

Integration of the continuity equation over the film thickness gives,

Equation:

$$
\begin{aligned}
0 &= \int_0^1 \left(\tfrac{1}{r}\,\tfrac{\partial}{\partial r}(r\,v_r) + \tfrac{\partial v_z}{\partial z}\right) dz \\
&= \int_0^1 \tfrac{1}{r}\,\tfrac{\partial}{\partial r}(r\,v_r)\,dz + v_z\big|_0^1 \\
&= \tfrac{1}{r}\,\tfrac{d}{dr}\left[r\int_0^1 v_r\,dz\right] - 1
\end{aligned}
$$

The velocity profile across the thickness is substituted into the above equation and the integration preformed.

Equation:

$$
\frac{1}{3r}\,\frac{d}{dr}\left(r\frac{dP}{dr}\right) + 1 = 0
$$

Integration and application of the boundary conditions give
Equation:

$$P = \frac{3}{4}\left(1 - r^2\right)$$

The pressure and radius can now be converted to dimensional variables so we can see the dependence of the parameters.
Equation:

$$P\left(r\right) = \frac{3\mu\, R^2\, U}{4h^3}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

The pressure distribution has a maximum at the center of the disk and decreases to zero at the outer radius of the disk. The pressure integrated over the area of the disk gives the force required to bring the disks together, each disk with a velocity U, when each disk is a distance h from the midplane.
Equation:

$$F = 2\pi \int_0^R P(r)\, r\, dr = \frac{3}{8}\frac{\pi\, \mu\, R^4\, U}{h^3}$$

This expression can be turned around to express the velocity of each disk approaching each other when a force F is applied.
Equation:

$$U = -\frac{dh}{dt} = \frac{8}{3}\frac{h^3\, F}{\pi\, \mu\, R^4}$$

This result is the classical Reynolds (1886) velocity for the thinning of two parallel disks.

If the applied force is constant, the above equation can be integrated to determine the time it takes to thin down from some initial thickness, h_i.
Equation:

$$\frac{1}{h^2} - \frac{1}{h_i^2} = \frac{16}{3} \frac{F\,t}{\pi\,\mu\,R^4}$$

It the initial thickness is large but unknown, then it can be assumed to be infinity with only a small error in the time to thin down to a small thickness. An explicit expression for the time to thin from infinite thickness to a thickness h is
Equation:

$$t = \frac{3}{16} \frac{\pi\,\mu\,R^4}{F\,h^2}$$

From this expression, we see that it will take an infinite time to thin to zero thickness. In reality, as the film becomes very thin, surface forces (disjoining pressure) will become important in accelerating or retarding the rate of thinning. If the surfaces are solid surfaces, contact will be made at high points (roughness) and the contact stresses may limit the thinning.

<u>Transient Drainage of a Vertical Film</u> Earlier we treated the steady film flow along an inclined plane. Here we will consider the transient drainage of a film that has zero flux at the upstream boundary. This corresponds to the transient behavior immediately after the flow if liquid is shut off in the problem of the steady flow along an incline plane. We will treat the wall as if it was vertical. If it is inclined from the vertical, the acceleration of gravity in the solution will need to be multiplied by the cosine of the angle from the vertical. It is assumed that the film is thin enough for the Reynolds number to be negligible and there are no ripples. Also, it is assumed that the thickness is large enough that surface forces (disjoining pressure) are negligible. The fluid in the film is assumed to be incompressible and Newtonian and the surrounding fluid is assumed to have zero density and viscosity. The surface tension and surface viscosity are neglected. The initial thickness of the film is assumed to be a constant value, h_i. Let z be

the coordinate in the direction of the film flow and x the direction perpendicular to the wall. The equations of motion and continuity equation for thin films, discussed in Chapter 6 have several terms that can be neglected. The resulting equations and boundary conditions follows.

Equation:

$$v_x = u, \quad v_z = v$$

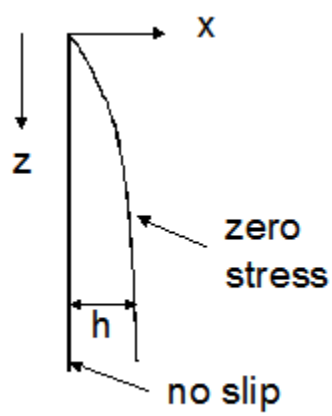
$$\left.\begin{aligned} 0 &= -\frac{\partial P}{\partial x} \\ 0 &= -\frac{\partial P}{\partial z} + \mu \frac{\partial^2 v}{\partial x^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} &= 0 \end{aligned}\right\}, \quad \begin{aligned} &t > 0 \\ &z > 0 \\ &0 < x < h(z,\, t) \\ &z \gg h \end{aligned}$$

$$h\,(z,\, 0) = h_i \qquad t = 0$$

$$v = 0 \qquad z = 0,\ t > 0$$

$$u = v = 0 \qquad x = 0$$

$$\left.\begin{aligned} u &= \frac{dx}{dt} = \frac{\partial h}{\partial t} + v\frac{\partial h}{\partial z} \\ \tau &= -\mu \frac{\partial v}{\partial x} = 0 \end{aligned}\right\} \qquad x = h\,(z, t)$$



The first equation states that there is zero potential gradient over the thickness of the film. Because the surrounding fluid has zero density and the surface tension is neglected, the pressure in the film is equal to that of the surrounding fluid. Thus,

Equation:

$$P = p - \rho\, g\, z = p_o - \rho\, g\, z$$
$$\frac{\partial P}{\partial z} = -\rho\, g$$

The velocity profile across the thickness of the film can be determined by integrating the second equation.

Equation:

$$v = \frac{\rho\, g\, h^2}{\mu}\left[\frac{x}{h} - \frac{1}{2}\left(\frac{x}{h}\right)^2\right]$$

The flux or flow rate per unit width of the film can be determined by integrating the velocity profile over the thickness of the film.

Equation:

$$\int_0^h v\, dx = \frac{\rho\, g\, h^3}{3\mu}$$

The continuity equation can be integrated over the thickness of the film.

Equation:

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\right) dx = \frac{\partial h}{\partial t} + v\frac{\partial h}{\partial z} + \int_0^h \frac{\partial v}{\partial z}\, dx$$

The derivative can be taken outside of the integral with the addition of another term that cancels the term in the previous equation.

Equation:

$$\int_0^h \frac{\partial v}{\partial z}\,dx = \frac{\partial}{\partial z}\int_0^h v\,dx - v\,\frac{\partial h}{\partial z}\bigg|_{x=h}$$

Thus the differential equation for the film thickness is
Equation:

$$\begin{aligned}
&\tfrac{\partial h}{\partial t} = -\tfrac{\rho\,g}{3\mu}\,\tfrac{\partial h^3}{\partial z} = -\tfrac{\rho\,g\,h^2}{\mu}\,\tfrac{\partial h}{\partial z}\\
&h\,(z,0) = h_i, \quad t = 0,\ z > 0\\
&h(0,t) = 0, \quad z = 0,\ t > 0
\end{aligned}$$

This is a first order, hyperbolic partial differential equation with constant initial and boundary conditions. Time and distance can be combined into a single similarity variable. The trajectories of constant values of the dependent variable can be calculated from the PDE.
Equation:

$$\begin{aligned}
dh &= 0 = \tfrac{\partial h}{\partial t}\,dt + \tfrac{\partial h}{\partial z}\,dz\\
\left(\tfrac{dz}{dt}\right)_{dh=0} &= -\frac{\frac{\partial h}{\partial t}}{\frac{\partial h}{\partial z}}\\
&= \tfrac{\rho\,g\,h^2}{\mu}, \quad h \le h_i
\end{aligned}$$

Since the origin of all changes in thickness occur at the origin, the equation can be integrated as straight-line trajectories for each value of thickness between zero and the initial condition.
Equation:

$$\left(\frac{z}{t}\right)_{dh=0} = \frac{\rho g h^2}{\mu}, \quad 0 < h < h_i$$

$$\left(\frac{z}{t}\right)_{h_i} = \frac{\rho g h_i^2}{\mu}$$

$$h(z,t) = \sqrt{\frac{\mu z}{\rho g t}}, \qquad \frac{z}{t} < \frac{\rho g h_i^2}{\mu}$$

$$h(z,t) = h_i, \qquad \frac{z}{t} > \frac{\rho g h_i^2}{\mu}$$

This is the classical solution for transient film drainage. Thick films initially drain very rapidly but the rate of drainage slows as the film thins.

Notice that where the thickness has thinned below the initial thickness, the thickness is independent of the value of the initial thickness. Also, notice that the solution does not have a characteristic time, length, or thickness. This suggests that the thickness, time and distance are self-similar. In fact these variables can be combined into a single variable.

Equation:

$$\frac{\rho g t h^2}{\mu z} = 1$$

or

$$\frac{t h^2}{z} = \frac{\mu}{\rho g}$$

for

$$h < h_i$$

The thickness normalized with respect to the initial thickness can be expressed as a function of a single similarity variable or if a system length is specified, it can be expressed as a function of the dimensionless distance and time.

$$
\begin{aligned}
h^* &= \frac{h}{h_i} \\
&= \begin{cases} \sqrt{\frac{\mu\, z}{h_i^2\, \rho\, g\, t}}, & \frac{\mu\, z}{h_i^2\, \rho\, g\, t} < 1 \\ 1, & \frac{\mu\, z}{h_i^2\, \rho\, g\, t} > 1 \end{cases} \\
&= \begin{cases} \sqrt{\eta}, & \eta < 1, \quad \eta = \frac{\mu\, z}{h_i^2\, \rho\, g\, t} \\ 1, & \eta > 1 \end{cases} \\
&= \begin{cases} \sqrt{\frac{z^*}{t^*}}, & ,\eta < 1, \quad z^* = \frac{z}{h_i}, \quad t = \frac{\rho\, g\, h_i\, t}{\mu} \\ 1, & \eta > 1 \end{cases}
\end{aligned}
$$

One may be interested in the volume of liquid that remains on a vertical wall of length L after the film is everywhere less than the initial thickness. This can be determined by integrating the film thickness profile over the length of the wall.

Equation:

$$
\begin{aligned}
&\int_0^L h\, dz = \sqrt{\frac{4\mu\, L^3}{9\rho\, g\, t}}, \quad \text{for } t > \frac{\mu\, L}{\rho\, g\, h_i^2} \\
&\int_0^1 h\, dz^* = \sqrt{\frac{4\mu\, L}{9\rho\, g\, t}}, \quad \text{where } z^* = z/L
\end{aligned}
$$

This expression shows that the amount of liquid remaining on a vertical wall is inversely proportional to the square root of time. This solution is valid only after the film has everywhere thinned below the initial thickness.

Assignment 9.1

Transient drainage of vertical film.

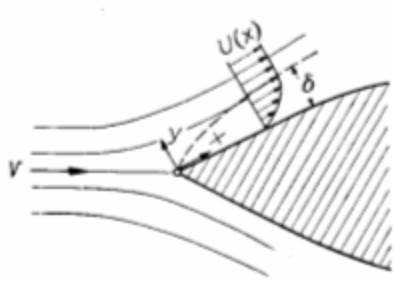
a. Combine the independent variables and parameters as a dimensionless similarity variable. Plot the normalized thickness as a function of the similarity variable.
b. Suppose the length of the system is L. Plot the profiles of the normalized thickness as a function of dimensionless distance for different values of dimensionless time, $(t = eps : 1 : 20)$.

Laminar Boundary Layer

The flow of an ideal fluid (inviscid and incompressible) in two dimensions can be calculated for many configurations through the use of potential flow and complex variables. At a solid boundary, the ideal fluid has zero flux (the normal component of velocity is equal to that of the solid) but the tangential velocity may not be equal to that of the solid. Real fluids have a finite viscosity and (with only a few exceptions) the tangential component of velocity is equal to that of the solid, i.e., the boundary condition of no-slip. In many cases, the "external flow" sufficiently far from a solid body can be modeled as that of an ideal fluid and the effect of finite viscosity effects the flow only near the surface of the body (and downstream of the body). These situations can be treated by application of the boundary layer theory.

The underlying assumption in the boundary layer theory is that there is a very thin layer near the body where the gradient of the tangential velocity is very large due to the action of viscosity and the no-slip boundary condition. Elsewhere the effect of viscosity is assumed to be unimportant and can be modeled as inviscid or potential flow.

The continuity equation and equations of motion were specialized in Chapter 6 with the assumption that the boundary layer thickness, δ , is small compared to the characteristic dimension of the body, L, i.e., $\delta/L << 1$. The equations (known as Prandtl's boundary layer equations) and boundary conditions are recalled here.



Boundary layer flow

along a wall
(Schlichting, 1960)

Equation:

$$\begin{aligned} v_x = v_y = 0, &\quad \text{at } \ y = 0 \\ v_x = U_\infty(x), &\quad y \to \infty \\ v_x = U_\infty(0), &\quad x = 0, \ y > 0 \end{aligned}$$

Equation:

$$\begin{aligned} \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \\ v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y^2}, \quad \delta << L \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \Rightarrow p = p\,(x), \quad \delta << L \end{aligned}$$

The remaining terms in the equation of motion and continuity equation are of similar magnitude if

$$V_y^o = O(U_\infty\,\delta/L), \qquad p^o = \rho\,U_\infty^2$$

$$\frac{L\nu}{\delta^2 U_\infty} = O\,(1) \quad \text{or} \quad \frac{\delta}{L} = O\left(\sqrt{\frac{\nu}{U_\infty L}}\right) = O\left(\frac{1}{\sqrt{\frac{\rho U_\infty L}{\mu}}}\right) = O\left(\frac{1}{\sqrt{\mathrm{Re}}}\right)$$

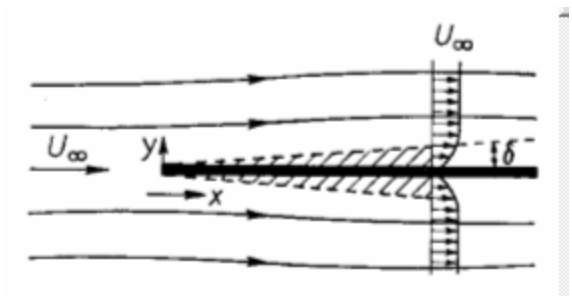
The assumption that the boundary layer thickness is small compared to the characteristic length of the body requires that the Reynolds number be large compared to unity if the terms in the equation of motion are to be of similar magnitude. If L is to represent the distance, x, from the leading edge of the boundary layer, the above relation describes the thickness of the boundary layer as a function of distance from the leading edge.

Equation:

$$\frac{\delta}{x} = O\left(\sqrt{\frac{\nu}{U_\infty x}}\right) = O\left(\frac{1}{\sqrt{\mathrm{Re}_x}}\right)$$

where

$$\mathrm{Re_x} = \frac{\rho U_\infty x}{\mu}$$



Flow past flat plate at zero incidence (Schlichting, 1960)

Another quantity that is of interest in boundary layer theory is the local drag coefficient due to the wall shear stress (some definitions differ by a factor of 2). The mean drag coefficient is the average value of this quantity over the surface of the body.

Equation:

$$C_f \equiv \frac{\tau_{xy}}{\rho {U_\infty}^2} = \frac{\mu}{\rho {U_\infty}^2} \frac{\partial v_x}{\partial y}\bigg|_{y=0}$$

<u>Laminar flow along flat plate</u>

The classical system for the study of laminar boundary layers is the flow of a fluid in uniform translation past a flat plate. The free stream velocity is constant and the pressure gradient is zero. The classical solution to this problem is the doctoral thesis of H. Blasius (1908). The equation of continuity is satisfied exactly by expressing the velocity as the curl of the

stream function. Since the system does not have a characteristic length, a similarity transformation makes it possible to combine the two independent variables (x, y) into a single independent variable, $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$. The equations reduce to a quasilinear third order ordinary differential equation for the dimensionless stream function. The solution is given as a series solution. Its derivation is tedious and will not be discussed here. The reader is referred to Schlichting (1960) for details.

An alternative approach is to keep the velocity components as the dependent variables and approximately satisfy the continuity equation by expressing the transverse velocity component in the equation of motion as follows.
Equation:

$$v_y = -\int_0^y \frac{\partial v_x}{\partial x}\, dy$$

This is the approach taken in BSL. They express the solution as a cubic polynomial in η.
Equation:

$$\eta = y/\delta, \qquad \delta(x) = 4.64\sqrt{\frac{\nu x}{U_\infty}}$$
$$\frac{v_x}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3, \quad 0 \le y \le \delta(x)$$

<u>Analogy with wall suddenly set in motion</u>

The previously mentioned solutions may be accurate solutions to the boundary layer equations but do not offer much physical insight. Here we will derive the solution to the boundary layer flow by using the flow due to a wall suddenly set in motion discussed in Chapter 8. Since the wall extends to infinity, there is no dependence on x and the equations of motion, initial condition, and boundary condition are as follows.
Equation:

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}, \quad y > 0,\ t > 0$$

Equation:

$$\begin{aligned} v_x &= 0, \quad t = 0,\ y > 0 \\ v_x &= U, \quad y = 0,\ t > 0 \\ v_x &= 0, \quad y \to \infty,\ t > 0 \end{aligned}$$

The solution derived by a similarity transform in Chapter 7 is,
Equation:

$$v_x = U\, erfc\left(\frac{y}{\sqrt{4\mu\, t/\rho}}\right)$$

The coordinates can be transformed such that the plate is stationary and the fluid is initially in uniform translation past the plate. The initial condition, boundary conditions, and solution are then as follows.
Equation:

$$\begin{aligned} v_x &= U_\infty, \quad t = 0,\ y > 0 \\ v_x &= 0, \quad y = 0,\ t > 0 \\ v_x &= U_\infty, \quad y \to \infty,\ t > 0 \\ v_x &= U_\infty\, erf\left(\frac{y}{\sqrt{4\nu\, t}}\right) \end{aligned}$$

This exact solution describes the diffusion of momentum from the uniformly translating fluid to the stationary plate. It describes growth of the boundary layer as a function of the similarity variable, $\eta = \frac{y}{\sqrt{4\nu\, t}}$. There is no convection of momentum because there is no dependence on x and the transverse component of velocity is zero.

Suppose now that the plate is not doubly infinite but only exists along the positive x axis and the flow is in the positive x direction.. Since there is now dependence on the x coordinate, boundary conditions depend on x and the convective terms in the equations of motion no longer vanish. Now consider the steady state flow past this semi-infinite plate. Assume that the flow is undisturbed until $x = 0$. The differential equations are the boundary layer equations with zero velocity gradients. The equations and boundary conditions are as follows.

Equation:

$$\begin{aligned}
&\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \\
&v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} \\
&v_x = v_y = 0, \quad \text{at } \ y = 0 \\
&v_x = U_\infty, \quad y \to \infty \\
&v_x = U_\infty, \quad x = 0, \ y > 0
\end{aligned}$$

Recall that the convection terms are the convective derivative of the x component of momentum. Thus the equation of motion can be expressed as follows.

Equation:

$$\begin{aligned}
&v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{Dv_x}{Dt} \\
&\frac{Dv_x}{Dt} = \nu \frac{\partial^2 v_x}{\partial y^2}
\end{aligned}$$

This equation can be used to describe the diffusion of momentum along a streamline which originated at $x = 0$ at $t = 0$. If the transverse velocity is zero, each streamline would be at a constant value of y. With steady flow, there is a one-to-one mapping between x and t along each streamline. However, this mapping is not the same for all streamlines because different streamlines have different velocities. Beyond the boundary layer, the velocity is the free stream velocity and at the surface of the plate, $y = 0$, the velocity is zero. We will make the assumption that the mapping for the

entire boundary layer can be approximated by using the average of the free stream velocity and the velocity at the plate.
Equation:

$$\begin{aligned} t\left(x\right) &= \tfrac{x}{(U_\infty+0)/2} \\ &= \tfrac{2x}{U_\infty} \end{aligned}$$

Also, we assume for the first approximation that the transverse velocity is zero such that each streamline is at a constant value of y. The diffusion equation now transforms into a parabolic PDE in x and y.
Equation:

$$\frac{Dv_x}{Dt} \approx \frac{Dx}{Dt}\frac{\partial v_x}{\partial x} = \frac{U_\infty}{2}\frac{\partial v_x}{\partial x} = \nu\frac{\partial^2 v_x}{\partial y^2}$$

This mapping of distance along the plate and time is substituted into the expression for the diffusion of momentum to a stationary plate that was introduced into a uniformly translating fluid at $t = 0$.
Equation:

$$v_x^{(1)} = U_\infty\, erf\left(\frac{y}{\sqrt{8\nu\, x/U_\infty}}\right)$$

This first approximation neglects the transverse velocity and assumes that the time since passage of the front of the plate corresponds to the average velocity of the fluid in the boundary layer. Although this solution is quite close to the exact, Blasius solution, it does not satisfy the continuity equation. We now derive the second approximation by application of the continuity equation.

$$\begin{aligned} v_y &= -\int_0^y \tfrac{\partial v_x}{\partial x}\, dy \approx -\int_0^y \tfrac{\partial v_x^{(1)}}{\partial x}\, dy \\ &= \tfrac{U_\infty}{\sqrt{U_\infty x/\nu}}\left[1 - \exp\left(\tfrac{-y^2}{8\nu x/U_\infty}\right)\right] \end{aligned}$$

The limiting value of the transverse velocity with this approximation is,

$$\lim_{y \to \infty} v_y = \frac{U_\infty}{\sqrt{U_\infty x/\nu}} = \frac{U_\infty}{\sqrt{\mathrm{Re}_x}}$$

This limiting value of the transverse velocity differs from the Blasius solution only by a coefficient of 0.865.

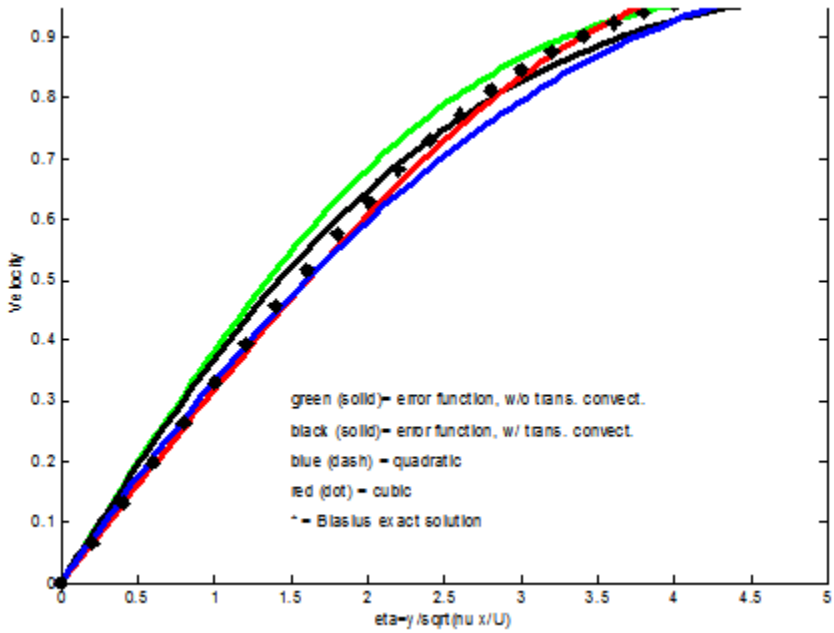
The transverse velocity results in convection of momentum away from the wall. If the convection velocity was constant, then its effect can easily be taken into account with the solution to the convection-diffusion equation. However, the transverse convection increases from zero at the wall to the limiting value in the free stream. Thus it makes more sense to use the average transverse velocity between the wall and at a point in the boundary layer for substitution into the solution of the convective-diffusion equation.

$$\begin{aligned}
\overline{v}_y &= \frac{\int_0^y v_y \, dy}{y} \\
&= \frac{U_\infty}{\sqrt{U_\infty x/\nu}} \left[1 - \frac{\sqrt{\pi}}{2} \frac{\sqrt{8\nu x/U_\infty}}{y} \mathrm{erf}\left(\frac{\mathrm{y}}{\sqrt{8\nu x/U_\infty}} \right) \right]
\end{aligned}$$

The second approximation will include the transverse convection by substituting the average transverse velocity into the convective-diffusion solution.

$$\begin{aligned}
v_x^{(2)} &= U_\infty \, \mathrm{erf}\left(\frac{y - \overline{v}_y x/U_\infty}{\sqrt{8\nu x/U_\infty}} \right) \\
&= U_\infty \, \mathrm{erf}\left\{ \frac{\mathrm{y}}{\sqrt{8\nu x/U_\infty}} - \frac{1}{\sqrt{8}} \left[1 - \frac{\sqrt{\pi}}{2} \frac{\sqrt{8\nu x/U_\infty}}{y} \mathrm{erf}\left(\frac{\mathrm{y}}{\sqrt{8\nu x/U_\infty}} \right) \right] \right\}
\end{aligned}$$

The first and second approximations are compared with the Blasius exact solution and quadratic and cubic approximation in the following figure.



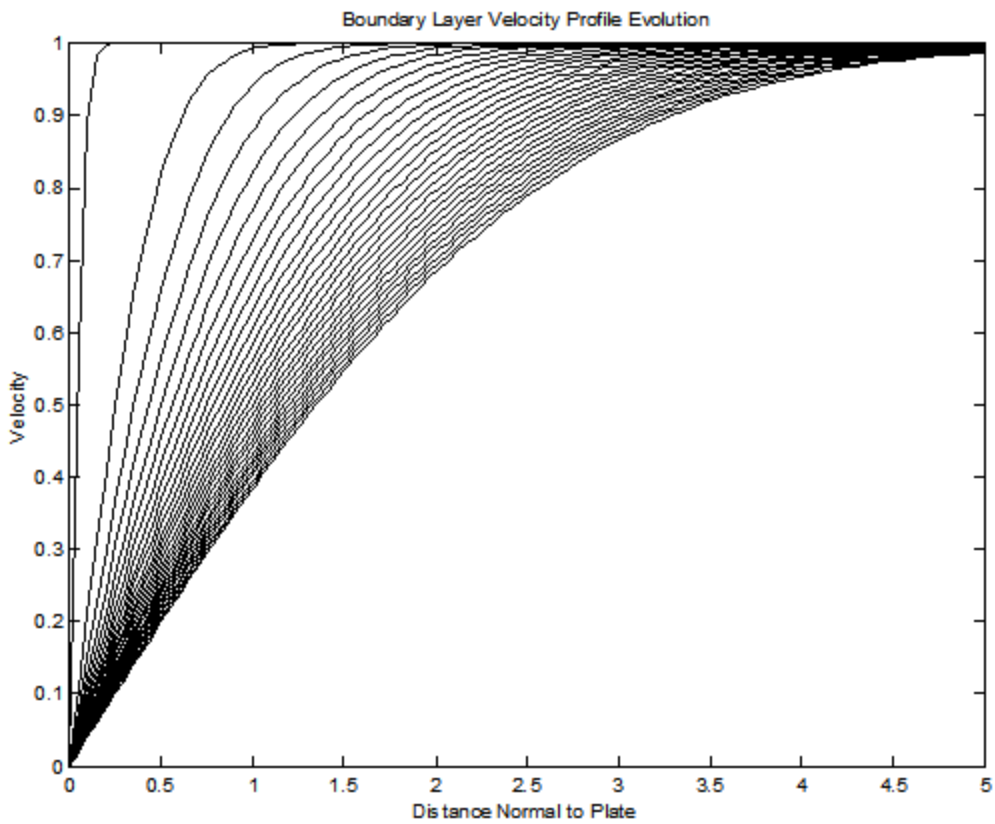
The inclusion of the transverse convection made an insignificant improvement in the velocity profile. Thus it will be neglected in the following. The drag coefficient is calculated from the first approximation.

Equation:

$$
\begin{aligned}
C_f &\equiv \frac{\tau_{xy}}{\rho {U_\infty}^2} = \frac{\mu}{\rho {U_\infty}^2} \frac{\partial v_x}{\partial y}\bigg|_{y=0} \\
&= \frac{2}{\sqrt{8\pi}} \frac{1}{\sqrt{U_\infty x/\nu}} \\
&= \frac{0.3989}{\sqrt{\mathrm{Re}_x}}
\end{aligned}
$$

This drag coefficient differs from the Blasius solution only by the coefficient of 0.332 in the exact solution.

The evolution of the boundary layer velocity profile with equal increments of distance from the leading edge of the plate is illustrated in the following figure. It is suggested that the student execute the *plate.m* file in the *boundary* directory to view the movie of the evolution of the velocity profile and to examine the equations used in the calculations.



Blasius solution for boundary layer flow past a flat plate

The previous solutions were instructive in that they illustrated the correspondence with the diffusion of motion from a plate to the bulk fluid. However, the approximate solutions did not exactly satisfy either the continuity equation or the equations of motion. Blasius used the stream function to exactly satisfy continuity equation. The equations of motions were simplified by the boundary layer assumption that the thickness of the boundary layer is small compared to the distance from the leading edge of the plate. Also, the pressure gradient is zero for this case.

Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$
$$y = 0: \quad u = v = 0$$
$$y \to \infty: \quad u = U_\infty$$

Assume that the dimensionless velocity profile can be expressed as a function of a similarity variable.
Equation:

$$\frac{u}{U_\infty} = u^* = u^*\left(\frac{y}{\delta(x)}\right)$$

The approximate solution had the following form:
Equation:

$$\frac{v_x^{(1)}}{U_\infty} = \mathrm{erf}\left(\frac{y}{\sqrt{8\nu\, x/U_\infty}}\right)$$

This suggests a similarity variable:
Equation:

$$\eta = \frac{y}{\sqrt{\nu\, x/U_\infty}} = y\sqrt{\frac{U_\infty}{\nu\, x}}$$
$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu\, x}}, \quad \frac{\partial \eta}{\partial x} = \frac{-\eta}{2x}$$

The equation of motion is expressed in terms of the stream function.
Equation:

$$u = \frac{\partial \psi}{\partial y}, \quad v = \frac{-\partial \psi}{\partial x}$$
$$\frac{\partial \psi}{\partial y}\,\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\,\frac{\partial^2 \psi}{\partial y^2} = \nu\,\frac{\partial^3 \psi}{\partial y^3}$$

Assume that the stream function is a function only of the similarity variable.
Equation:

$$\psi = \psi(\eta)$$
$$u = \frac{\partial \psi}{\partial y} = \frac{d\psi}{d\eta}\,\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}\,\frac{d\psi}{d\eta}$$

Make dimensionless:
Equation:

$$U_\infty\, u^* = \sqrt{\frac{U_\infty}{\nu\, x}}\,\psi_o\,\frac{d\psi^*}{d\eta}$$
$$u^* = \frac{\psi_o}{\sqrt{\nu\, U_\infty\, x}}\,\frac{d\psi^*}{d\eta}$$
$$\Rightarrow \psi_o = \sqrt{\nu\, U_\infty\, x}$$

The dimensionless stream function is expressed as a function of only the similarity variable.
Equation:

$$\begin{aligned}
\psi^* &= f(\eta) \\
\psi &= \sqrt{\nu\, U_\infty\, x}\, f(\eta) \\
u &= U_\infty\,\frac{df}{d\eta} = U_\infty\, f' \\
v &= \frac{-\partial \psi}{\partial x} \\
&= -\frac{\partial}{\partial x}\left[\sqrt{\nu\, U_\infty\, x}\, f(\eta)\right] \\
&= \frac{1}{2}\sqrt{\frac{\nu\, U_\infty}{x}}\,\left[\eta\, f' - f\right] \\
\frac{\partial u}{\partial x} &= -\frac{U_\infty}{2x}\,\eta\, f'' \\
\frac{\partial u}{\partial y} &= U_\infty\sqrt{\frac{U_\infty}{\nu\, x}}\, f'' \\
\frac{\partial^2 u}{\partial y^2} &= \frac{U_\infty^2}{\nu\, x}\, f'''
\end{aligned}$$

Substituting the above equations into the equation of motion and cancellation of two terms results in the following ordinary differential equation.

Equation:

$$
\begin{aligned}
&2f''' + f\,f'' = 0 \\
&f(0) = f'(0) = 0 \\
&f'(\eta \to \infty) = 1
\end{aligned}
$$

This is a third order ODE with two conditions at $\eta \to \infty$ and one condition at . It is convenient to solve it as a set of first order ODEs with initial conditions, two of which are specified and the third adjusted such as to satisfy the condition at infinity.

Equation:

$$
\mathbf{Y} = \begin{bmatrix} f \\ f' \\ f'' \end{bmatrix}, \quad \frac{d\mathbf{Y}}{d\eta} = \begin{bmatrix} f' \\ f'' \\ f''' \end{bmatrix} = \begin{bmatrix} f' \\ f'' \\ -f\,f''/2 \end{bmatrix} = \begin{bmatrix} Y_2 \\ Y_3 \\ -Y_1\,Y_3/2 \end{bmatrix}
$$

This set of ODEs can be solved numerically by one of the ODE solvers and the initial value of f'' iterated until the boundary condition at infinity is matched. The code of this calculation is in the boundary directory as files, *blasius.m*, *blasiusf.m*, and *balsiusd.dat*.

Assignment 9.2: Boundary layer flow past a wedge

1. Derive the equations for boundary layer flow past a wedge. Use a factor of in the denominator of the similarity variable to be in keeping with contemporary textbooks.
2. Use the code in the boundary directory of the CENG 501 website to solve the Flakner-Skan equation.
3. Plot the velocity profiles as a function of the similarity variable for different angles of the wedge relative to the approaching free-stream

velocity. Replace the parameter beta with the angle in degrees.
4. Illustrate the boundary layer thickness by plotting contour lines of 10
5. Plot the equally spaced streamlines for the same cases.

Assignment 9.3: Flow in a wedge with zero shear at $\theta = 0$.
Start from the continuity and Navier-Stokes equation and derive the equations for the flow field near a corner for flow in a wedge of fluid with no slip on one side and zero shear stress along $\theta = 0$. List all your assumptions.

1. Derive expressions for the stream function, velocity, and pressure.
2. For what distance from the corner is the solution valid?
3. What normal stress is required to keep the = 0 surface flat?
4. If the surface of zero shear stress can sustain only finite normal stress, in which way will the surface deform? Recognize that the no-slip surface can travel in either direction.
5. After deriving the equations, view and plot the flow field for various angles using wedge.m file in the creeping directory of the CENG 501 website.

Assignment 9.4: Rise of a spherical, inviscid bubble in a liquid.
Start from the continuity and Navier-Stokes equation and derive the equations for the flow field of a spherical, inviscid bubble rising in a liquid by buoyancy. List all assumptions.

1. Derive expressions for the stream function, velocity, and pressure.
2. Derive the expression for the terminal rise velocity. How does it differ from the case with no slip?

Flow with Free Surface

We have already encountered free surfaces in systems such as the drainage of a liquid along a wall. In this case the free surface was a material surface and the boundary condition was that of continuity of pressure and shear stress. The same boundary conditions would be used for wind-driven waves on water and the shape of the vortex formed when water drains from a bathtub. The dimensionless numbers of importance are the Reynolds number $N_{\mathrm{Re}} = \rho\, U\, L/\mu$, Froude number $N_{Fr} = U^2/g\, L$, and gravity number $N_G = \frac{\rho\, g\, L^2}{\mu\, U} = \frac{N_{\mathrm{Re}}}{N_{Fr}}$. These mentioned systems are of a macroscopic scale compared to surface forces and rheology and thus surface tension, surface elasticity, and surface viscosity were not significant. However, when the system dimensions become about 1 cm or less surface forces are no longer negligible and play an important role in the shape of the interface and in transport processes. The capillary number $N_{Ca} = \mu\, U/\sigma$ and Bond number $N_{Bo} = \rho\, g\, L^2/\sigma$ introduced in Chapter 6 become important dimensionless groups that quantify the ratio of viscous/capillary and gravity/capillary forces. As the dimensions decrease to about 1 mm we are in the range of capillary phenomena where surface tension and contact angles become important (e.g., the rise of a wetting liquid in a small capillary). As the dimensions decrease to $1\, \mu m$ we are in the colloidal regime and not only is capillarity a dominant effect but also particles have spontaneous motion due to Brownian motion and thin films display optical interference as in the color of soap films. When the dimensions decrease to the range of 1 nm, it is necessary to include surface forces due to electrostatic, van der Waals, steric and hydrogen bonding effects to describe the thermodynamics and hydrodynamics of the fluid interfaces. At this scale the phases can no longer be assumed to be homogenous right up to the interface. The overlap of the inhomogeneous regions next to the interfaces results in forces that either attract or repel the interfaces.

Boundary Conditions at a Fluid-Fluid Interface

Analysis of macroscopic systems usually assume the fluid-fluid interface to simply be a surface of discontinuity in the density and viscosity of the bulk phases with no discontinuity in stress, (i.e., continuous pressure and shear stress across the surface). If there is no significant mass transfer, the surface is also a material surface and thus follows the motion of the fluid particles at the surface.

Systems with a length scale about a centimeter or less and having fluid-fluid interfaces can no longer neglect the discontinuity in stress across the interface. A momentum balance across the interface is needed to describe the stress at the

boundary. Also, if the system has surface-active components that affect the surface tension and/or surface viscosity, then a material balance is also needed to determine the composition of the interfacial region. A general treatment of the momentum balance at fluid-fluid interfaces is given in Chapter 10 of Aris, *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. We summarize the results here assuming no slip at the surface and a Newtonian surface constitutive relation. The terms in the momentum balance are given on the left side and its description is given on the right side.

$\gamma \frac{dU^i}{dt} =$	inertial force
$+\gamma g^i$	body force
$+\left[\left[T^{ij}\right]\right] n_j$	traction of the bulk phases
$+n^i 2H\sigma$	normal force due to surface tension
$+t^i_\alpha \, a^{\alpha\beta}\sigma_{,\beta}$	gradient of surface tension
$+\left(\kappa+\varepsilon\right)t^i_\alpha \, a^{\alpha\beta}\left(a^{\lambda\mu}t^j_\lambda U_{j,\mu}\right)_{,\beta}$	dilatational force
$+2\varepsilon \, K \, t^i_\alpha \, a^{\alpha\beta}t^j_\beta U_j$	force due to $\mathrm{U_s}$ and total curvature
$-\varepsilon \, t^i_\alpha \, \varepsilon^{\alpha\beta}\left[\varepsilon^{\lambda\mu}\left(t^j_\mu \, U_j\right)_{,\lambda}\right]_{,\beta}$	$\mathrm{curl}(\mathrm{curl}U_s),$ surface shear force
$-2\varepsilon \, t^i_\alpha \, \varepsilon^{\alpha\lambda}b_{\lambda\beta} \, \varepsilon^{\beta\mu}\left(n^j U_j\right)_{,\mu}$	effect of varing normal velocity
$+n^i 2H \left(\kappa+\varepsilon\right)\left(t^j_\lambda \, a^{\lambda\mu} \, U_{j,\mu}\right)$	normal force due to dilatation
$+n^i 2\varepsilon \, t^j_\lambda \, \varepsilon^{\lambda\alpha} \, b_{\alpha\beta} \, \varepsilon^{\beta\mu} \, U_{j,\mu}$	normal force due to shear

The surface constitutive equation for a Newtonian interface is (Slattery, 1990)

$$\mathbf{T^s} = [\sigma + (\kappa+\varepsilon)\nabla_s \bullet \mathbf{v^s}]\mathbf{I^s} + 2\varepsilon\mathbf{e_s}$$

The surface properties are a function of the composition of the interface. The species mass balance at the interface is given as (Edwards, et al., 1991)
Equation:

$$\frac{\partial \Gamma_n}{\partial t} + \nabla_s \bullet (\mathbf{v^s} \, \Gamma_n) + \nabla_s \bullet (\mathbf{I_s} \bullet \mathbf{j}^s_n) = R^s_n + \mathbf{n} \bullet [\![\mathbf{j}_n]\!]$$

The parallel between the mass and momentum balances and constitutive equations for interfaces and bulk fluids should be noted (Gurmeet Singh, 1996). This analogy with bulk fluids is more complete if a *surface pressure* due to the reduction of the surface or interfacial tension due to adsorption is defined.

Equation:

$$\pi = \sigma_o - \sigma(\Gamma)$$

The surface properties are as follows.

Symbol	Property
γ	mass per unit area
Γ_n	surface excess concentration of species n
σ	surface or interfacial tension
π	surface pressure
κ	surface dilatational viscosity
ε	surface shear viscosity
$a^{\alpha\beta}\, b_{\alpha\beta}\, t^i_\alpha$	surface tensors
$\varepsilon^{\alpha\beta}$	surface permutation symbol
$U^i, \quad \mathbf{v^s}$	surface velocity
$\mathbf{j}^s_n$	surface diffusional flux
H	mean curvature
K	Gaussian or total curvature

The normal component of stress. The discontinuity in the normal component of the total stress tensor for a hydrostatic system is as follows.
Equation:

$$[\mathbf{T}]_{\mathbf{n}} = [p]\mathbf{n} = 2H\,\sigma\,\mathbf{n}$$

where H is the mean curvature of the surface and σ is the surface or interfacial tension. The mean curvature can be expressed as a function greatest and least curvatures of curves in the surface k_1 and k_2 or the *principal curvatures* in the directions of the principal curvature.
Equation:

$$2H = \kappa_1 + \kappa_2$$

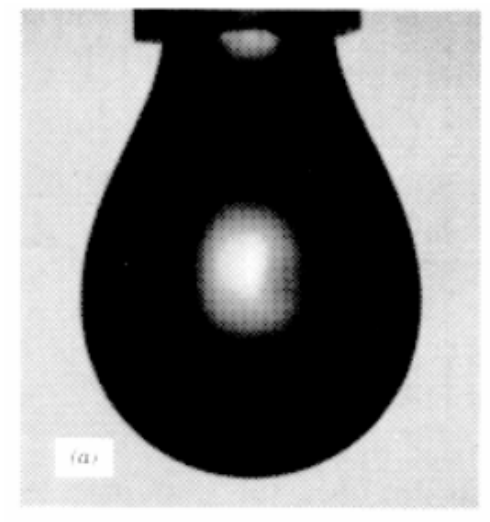
The curvature in one of the principal directions can be expressed in terms of the equation for the arc of the curve. Suppose one principal direction is in the x-y plane.
Equation:

$$\begin{aligned} y &= h\,(x), \quad h' = \tfrac{dh}{dx}, \quad h'' = \tfrac{d^2h}{dx^2}, \quad ds = \pm\sqrt{(dx)^2 + (dy)^2} \\ \phi &= \arctan\,(h') \\ \kappa &\equiv \tfrac{d\phi}{ds} \\ &= -\tfrac{d\cos\phi}{d\,h} \\ &= \frac{h''(x)}{\left\{1+[h'(x)]^2\right\}^{3/2}} \end{aligned}$$

Thus the curvature can be determined from the coordinates of the surface. We see that when the slope is small, the curvature can be approximated by the second derivative. A surface that is translational symmetric has zero curvature in one direction (e.g., surface of a cylinder). The two principal curvatures are equal on the surface of a sphere. A saddle shaped surface can have zero curvature because the principal curvatures have opposite signs.

The difference in pressure across a curved interface is called the *Laplace pressure* after the Laplace-Young equation. This is a classical equation used to determine

the shape of a static meniscus or to determine the surface tension from the shape of a static meniscus such as the pendant drop shown here. This is a drop of water suspended from the tip of a capillary tube surrounded by air. Water surrounded by oil would be similar. Suppose we let the origin our coordinate system be the bottom of the drop. The system is an axisymmetric and has two radaii of curvature.



Pendant water drop in air
(Adamson, 1990)

Equation:

$$2H = \kappa_1 + \kappa_2 = \frac{z''}{\left(1+z'^2\right)^{3/2}} + \frac{z'}{x\left(1+z'^2\right)^{1/2}} = \frac{d\phi}{ds} + \frac{\sin\phi}{x}$$

The system is hydrostatic so the pressure profile can be expressed as the pressure jump at the origin and the difference in hydrostatic pressure along the profile.
Equation:

$$[p] = \Delta p_o - \Delta\rho\, g\, z$$
$$\Delta p_o = 2\sigma/r_o$$
$$[p] = 2\sigma/r_o - \Delta\rho\, g\, z$$

where Δp_o Laplace pressure at the apex of the drop and r_o is the radius of curvature of the apex of the drop.

The surface or interfacial tension is found using the pendant drop analysis by estimating the value of the tension that best fits the calculated meniscus shape with the shape captured in a video image.

When the pendent drop apparatus is designed so the meniscus is *pinned* to either the inner or outer edge of the tip of the needle, the contact angle does not influence the shape of the meniscus. If the drop rests on a flat surface, it is called a sessile drop and the elevation of the profile from the surface is a function of the contact angle that the meniscus makes with the substrate.

A characteristic length scale can be determined for the hydrostatic meniscus. Suppose that the datum of elevation corresponds to the apex of the drop. The hydrostatic profile is described by the following dimensionless Young-Laplace equation.

Equation:

$$\frac{\sigma}{L}\frac{2}{(r_o/L)} - \Delta\rho\, g\, L\, z^* = \frac{\sigma}{L}\left(\kappa_1^* + \kappa_2^*\right) = \frac{1}{L}\left(\frac{d\phi}{ds^*} + \frac{\sin\phi}{x^*}\right)$$

$$\frac{2}{B} - \left[\frac{\Delta\rho\, g\, L^2}{\sigma}\right] z^* = \left(\kappa_1^* + \kappa_2^*\right) = \frac{d\phi}{ds^*} + \frac{\sin\phi}{x^*}$$

where

$$s^* = \frac{s}{L}, \quad x^* = \frac{x}{L}, \quad z^* = \frac{z}{L}, \quad \kappa^* = L\,\kappa, \quad B = \frac{r_o}{L}$$

If the dimensionless group is specified to equal unity, a characteristic length is defined.

Equation:

$$L = \sqrt{\frac{\sigma}{\Delta\rho\, g}}$$

This characteristic length, called the *capillary* constant, is sometimes defined with a factor of 2 multiplying the surface tension. It has a value of 2.7 mm for the water-air interface at ambient conditions. This length is representative of the meniscus height of water next to a vertical wall that is wetted by water.

The dimensionless differential equations are now as follows.

Equation:

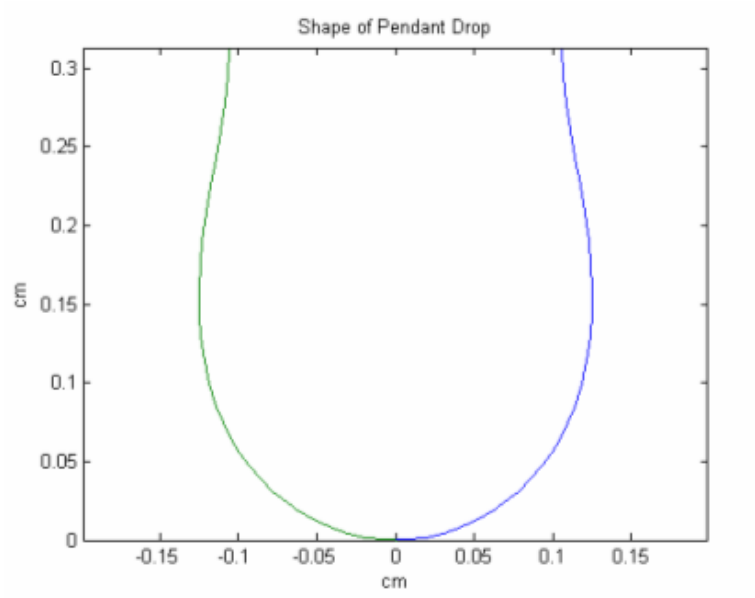
$$\frac{d\phi}{ds^*} = \frac{2}{B} - z^* - \frac{\sin\phi}{x^*}$$
$$\frac{dz^*}{ds^*} = \sin\phi$$
$$\frac{dx^*}{ds^*} = \cos\phi$$

Equation:

$$\left.\begin{aligned} \phi &= 0 \\ z^* &= 0 \\ x^* &= 0 \end{aligned}\right\} \quad \text{at} \quad s^* = 0$$

The system of ODEs is computed from the apex of the drop where the dimensionless radius of curvature at the apex is a parameter, B. The value of B is adjusted until the best fit of the measured drop shape is obtained.

Assignment 11.1
Estimation of surface tension. Estimate the surface tension of a dilute surfactant solution, given the shape of a pendant drop shown here and data stored in the file, *shape.dat* in the Surface directory. You may use the code in *pendant.m* in the *Surface* directory to generate the dimensionless pendant drop profiles. Use *units.m* to plot the drop shape. Note: The program is not automated. *Pendant.m* computes the dimensionless shape and *units.m* converts it to cgs units to compare with the measured profile.



Surface tension gradients. If the system has only two components, i.e., the components comprising each phase then the surface or interfacial tension and contact angle is all that is required to describe the surface effects. However, if the system has another component that is surface active as to adsorb at the interface and reduce the surface or interfacial tension, then the interface must be treated as a two dimensional phase for which a mass and momentum balance is required. The mass per unit area of the surface-active component is the surface excess concentration or the amount adsorbed. If the system is not at equilibrium, i.e., not hydrostatic, then concentration gradients may exist in the interface that result in surface tension gradients in the interface. The difference between the clean interface tension and the local tension is called the *surface pressure*. The gradients in the surface pressure contribute to tangential stresses in the interface. It has the same role in the surface momentum balance as pressure gradient in the momentum balance for three-dimensional flow. For example, as liquid drains from a soap film, the drag of the liquid on the interface stretches the interface. The resulting expansion of the interface reduces the surface concentration of soap on the interfaces. This establishes a surface tension or surface pressure gradient between the interface in the film and the interface in the meniscus. This gradient tends to oppose the motion of the interface and thus tends to maintain the interface immobile as the liquid drains from between the two near-immobile interfaces. At the same time this surface tension gradient tends to pull the interface from the meniscus back into the film. This leads to a turbulent motion of the interface at the boundary between the meniscus and the film. This effect called "marginal regeneration" is a *Marangoni* effect caused by the surface tension gradient.

Plate IV. An irregular mobile film in a vertical rectangle

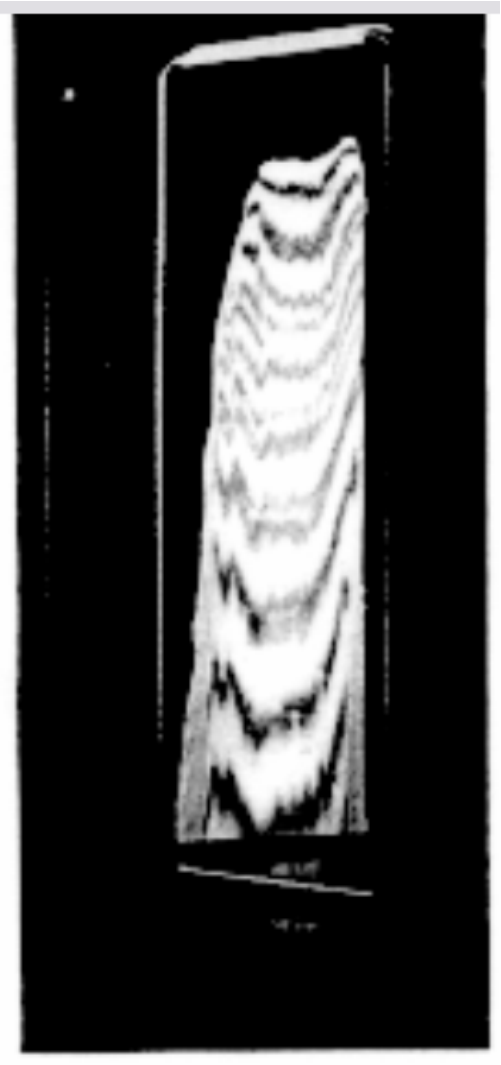


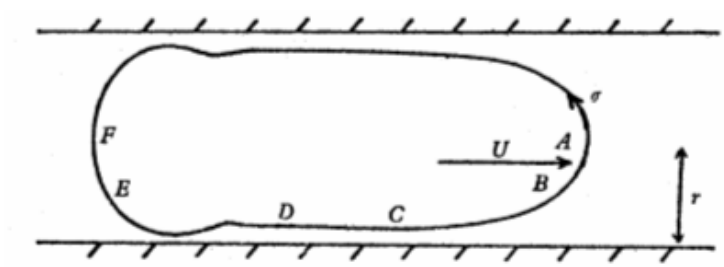
Plate I. Rigid film in a rectangular vertical

Surface viscosity. Adsorption of a surface-active component at an interface not only changes the surface tension or surface pressure but can also affect the surface rheology. Material adsorbed at interfaces form two-dimensional surface phases that may be gasous, expanded liquid, condensed liquid, or solid. The surface viscosity can change by more than a order of magnitude at a transition from one surface phase to another. This is analogous to the change in viscosity of bulk fluids at phase transitions. The attached figure shows vertical soap film drainage of a system is similar to that of the mobile film except that dodecanol was added to the sodium dodecyl sulfate (SDS) solution. The dodecanol screens the electrostatic

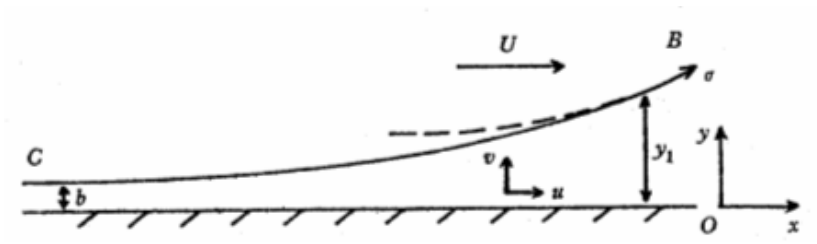
repulsion of the SDS at the interface and promote the formation of a condensed liquid monolayer. This monolayer is rigid in this system and the films drains much more slowly than in the case of the mobile film. The mechanism of this difference in the drainage of foam films has been explained in terms of the surface tension gradient driven instability and the stabilizing effect of a large surface viscosity (Joye, et al., 1994, 1996).

Film Drainage and Deposition with Laplace Pressure

In Chapter 8 we modeled the gravity drainage of a film along a wall neglecting the pressure between the liquid and gas because the mean curvature of the system was very small compared to the length of the film. Now suppose that we have a film that in connected to a curved meniscus. The meniscus may be moving along a substrate and depositing a film or gathering up a deposited film, e.g., a bubble in a capillary tube. Alternatively, the substrate may be stationary with respect to the meniscus and the film is draining into the mensicus, e.g. foam or emulsion film between two bubbles or drops. For simplicity, we will assume that we have a pure system so there are no surface tension gradients or effects of surface viscosity. We will assume that the system is translational invarient. A schematic of some possible system configurations are shown below.

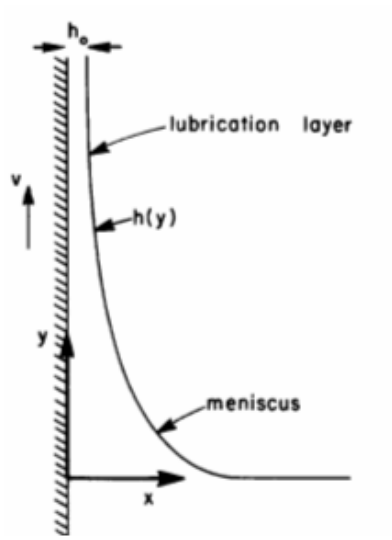


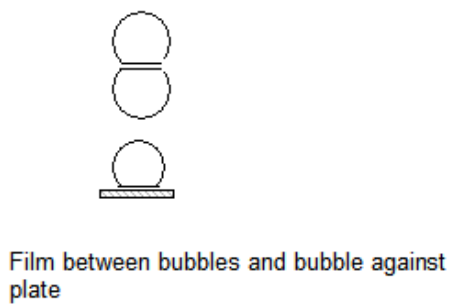
Section of a bubble in a horizontal tube.



The transition region.

Configration of a bubble in a tube (Breatherton, 1961)





The continuity equation and equations of motion were specialized for lubrication and film flow in Chapter 6. The equations to $O(h_o/L)$ or $O(h_o/L)2$ are as follows.

Equation:

$$\frac{\partial(h\,\overline{v}_{12})}{\partial x_{12}} + \frac{\partial h}{\partial t} = 0$$

$$0 = -\nabla_{12}P + \mu\frac{\partial^2 v_{12}}{\partial x_3^2}, \quad x_3 < h$$

$$0 = -\frac{\partial P}{\partial x_3}, \qquad x_3 < h$$

$$P = p - \Delta\rho\, g\, z$$

The systems with a solid substrate will have the boundary condition of no-slip at the solid boundary and zero shear stress at the pure-fluid interface. In the case of two bubbles or drops coming into contact, the mid-plane is a plane of symmetry and has zero shear stress. It will be assumed the fluid interface is immobile in this latter case. The variable, h, is the half-film thickness in this case. Since this case has zero shear on one surface and no slip on the other surface, the solution will be derived as for the cases with the solid substrate. The boundary conditions are as follows.

Equation:

$$v_{12} = \begin{cases} 0 & \text{at } x_3 = 0, \quad \text{for stationary substrate or bubble} \\ \pm U & \text{at } x_3 = 0, \quad \text{for substrate moving relative to meniscus} \end{cases}$$

$$\frac{\partial v_{12}}{\partial x_3} = 0 \quad \text{at } x_3 = h$$

The pressure is uniform across the thickness of the film so the velocity profile can be determined by integrating the equation of motion over the film thickness and applying the boundary conditions.

Equation:

$$v_{12} = \frac{\nabla P}{\mu}\left(\frac{x_3^2}{2} - h\,x_3\right) + \begin{cases} 0 \\ \pm U \end{cases}$$

$$\overline{v}_{12} = -\frac{h^2}{3\mu}\nabla P + \begin{cases} 0 \\ \pm U \end{cases}$$

The average velocity is substituted into the equation of continuity.

Equation:

$$\frac{\partial h}{\partial t} = \frac{1}{3\mu}\frac{\partial\left(h^3 \nabla_{12} P\right)}{\partial x_{12}} - \begin{cases} 0 \\ \pm \frac{\partial h}{\partial x_{12}} U \end{cases}$$

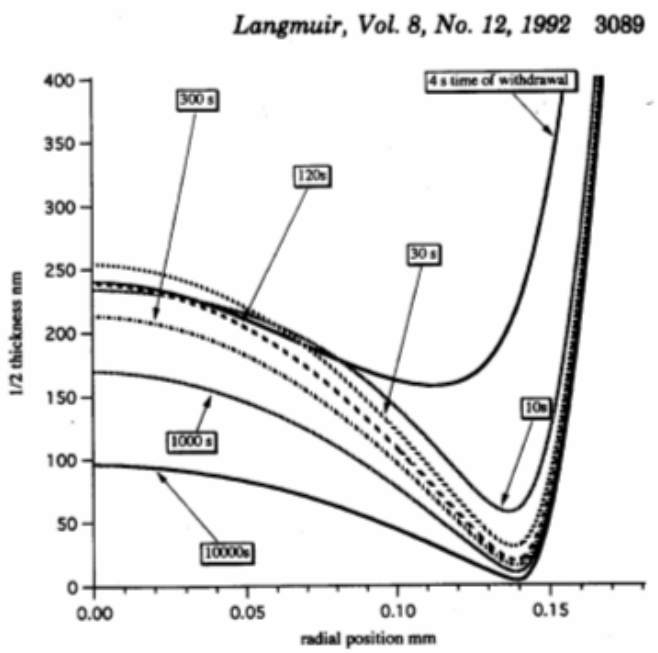
The pressure can be expressed in terms of the thickness by application of the Young-Laplace equation assuming that the system has no dependence on x_2.

Equation:

$$P = \frac{\sigma\, h''(x)}{\left\{1 + [h'(x)]^2\right\}^{3/2}} + \Delta\rho\, g\, z$$

Substituting the pressure into the previous equation results in a forth order quasilinear partial differential equation for the film thickness. These equations can be solved directly by numerical simulation. The equations will be specialized for special cases.

<u>Drainage of foam film.</u> Deriving the equations in cylindrical coordinates and dropping the terms that originated from the substrate velocity and gravity can describe the drainage of the thin, horizontal, circular film between two bubbles. If the interface remains plane-parallel, the Reynolds film drainage model in Chapter 9 can describe the drainage. The shape if the interface described by the above equations is "dimpled" and was investigated by Joye, et al., (1992). The pressure gradient in the film would be zero if the film were flat. The change in curvature where the film merges into the meniscus results in a large pressure gradient. This pressure gradient drains liquid from this region and this drainage causes a local thinning of the film. This leaves a thicker film or "dimple" in the center of the film. This axisymmetric drainage is unstable if the surface shear viscosity is small and the dimple will slip out to one side (Joye, et al., 1994, 1996).



Film profile: $R_c = 1.8mm$,
$Q = 10^{-10}m^3/8$, $\gamma = 30dyn/cm$,
disjoining pressure not included.

Bubble in a capillary. The motion of a bubble in a small capillary was investigated by Bretherton in 1961. With steady state, absence of gravity, and small slope in the film, the differential equations for the thickness of the film simplify to
Equation:

$$\frac{d^3h}{dx^3} = \frac{3\mu U}{\sigma}\frac{h-b}{h^3}$$

where b is the asymptotic thickness of the film that is left on the wall far from the meniscus. Near the front of the film where the thickness is very large compared to the asymptotic thickness, the right-hand side of the above equation can be approximated by zero. The general solution in this region for some constants A, B, and C is as follows.
Equation:

$$\frac{d^3h}{dx^3} \approx 0, \quad h >> b$$
$$h \cong \frac{1}{2}A\left(\frac{3\mu U}{\sigma}\right)^{2/3}\frac{x^2}{b} + B\left(\frac{3\mu U}{\sigma}\right)^{1/3}x + Cb$$

This thick portion of the film merges with the spherical cap at the front of the bubble. Thus the asymptotic film thickness, b, can be determined by requiring this portion of the film to have the same mean curvature as the front of the bubble.
Equation:

$$2H \cong \frac{d^2h}{dx^2} + \frac{1}{R} \cong \left(\frac{3\mu U}{\sigma}\right)^{2/3}\frac{A}{b} + \frac{1}{R}, \quad \text{near front of film}$$
$$2H \cong \frac{2}{R}, \qquad \text{in front of bubble}$$
$$\text{thus}$$
$$\frac{b}{R} \approx A\left(\frac{3\mu U}{\sigma}\right)^{2/3}$$

The constants of integration of the general solution of this film profile was determined matching with the shape of the meniscus at the front of the bubble with the solution by numerical integration. The asymptotic thickness of the film deposited by the advancing meniscus thus was found to be
Equation:

$$\frac{b}{R} = 0.643\left(\frac{3\mu U}{\sigma}\right)^{2/3}$$

where R is the mean radius of curvature of the spherical cap at the front of the bubble.

In the thin film region, CD, the film thickness is approximately equal to the asymptotic thickness and the differential equation can be linearized.
Equation:

$$\frac{d^3h}{dx^3} \cong \frac{3\mu U}{\sigma}\,\frac{h-b}{b^3}$$
$$\text{with the general solution}$$
$$\frac{h}{b} = 1 + \alpha\, e^{\xi} + \beta\, e^{-\frac{1}{2}\xi}\cos\left(\sqrt{3}\xi/2\right) + \gamma\, e^{-\frac{1}{2}\xi}\sin\left(\sqrt{3}\xi/2\right)$$
$$\xi = \left(\frac{3\mu U}{\sigma}\right)^{1/3}\frac{x}{b}$$

The asymptotic film thickness is approached if the dimensionless length of the bubble is large compared to unity. It is assumed that this is the case. Different parts of the general solution apply to the front and back of the bubble.
Equation:

$$\frac{h}{b} - 1 = \alpha\, e^{\xi}, \qquad \text{near front of bubbl}\,e$$
$$\frac{h}{b} - 1 = \beta\, e^{-\frac{1}{2}\xi}\cos\left(\sqrt{3}\xi/2\right) + \gamma\, e^{-\frac{1}{2}\xi}\sin\left(\sqrt{3}\xi/2\right), \quad \text{near rear of bubble}$$

The integration constant for the front part of the film, α , can be set to unity by a suitable choice of the origin of the axis. One integration constant for the rear of the bubble can be set to unity by suitable choice of the origin of the coordinate axis but the other must be specified to match the thickness of the film that is approaching the rear of the bubble. If this thickness is the asymptotic thickness, *b*, then there is

a unique solution for the film at the back of the bubble. These profiles shows oscillations in thickness as the rear of the bubble is approached. If the approach thickness is b, then the minimum thickness is $0.716b$. The capillary suction at the back of the bubble results in a local thinning of the film similar to the local thinning that occurs in the foam film between two bubbles.

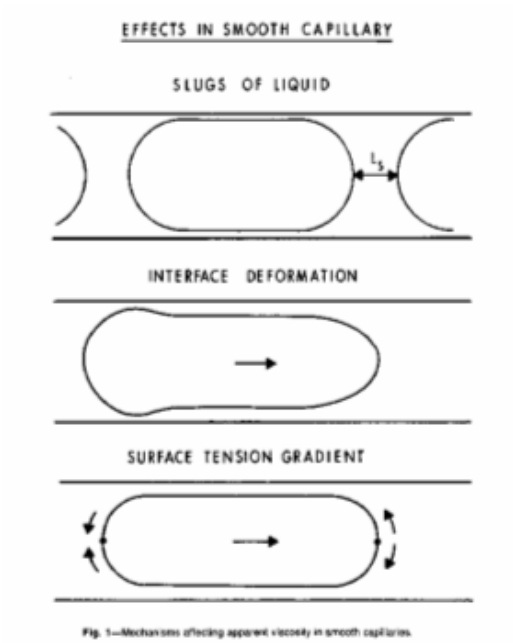
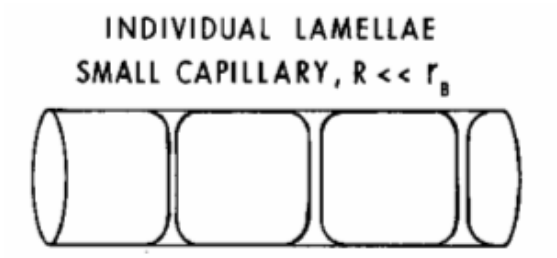
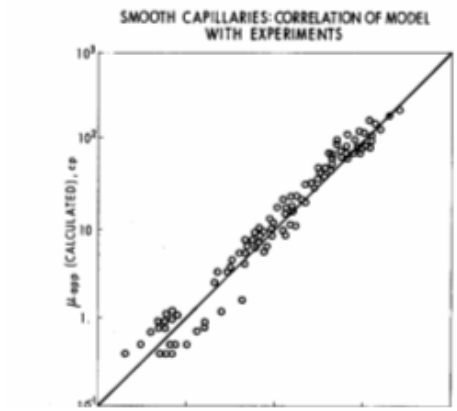


Fig. 1—Mechanisms affecting apparent viscosity in smooth capillaries.

Under static conditions, the Laplace pressures at the front and back of the bubble are both equal to $2\sigma/R$ with opposite sign, so the net pressure drop is zero. Under dynamic conditions the film profiles and thus the curvature at the front and back of the bubble are different. The dissimilar shapes of the front and back of the bubble results in a net dynamic pressure drop across the bubble.

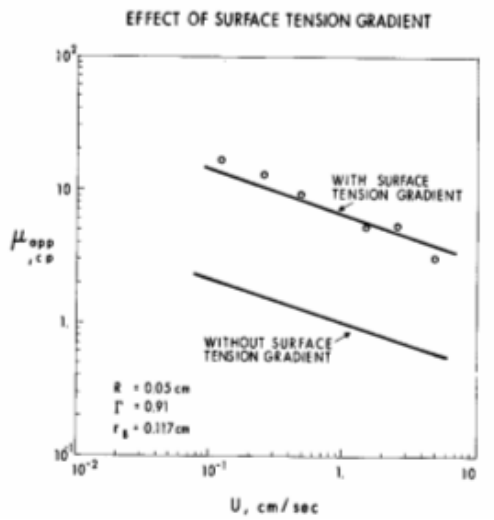


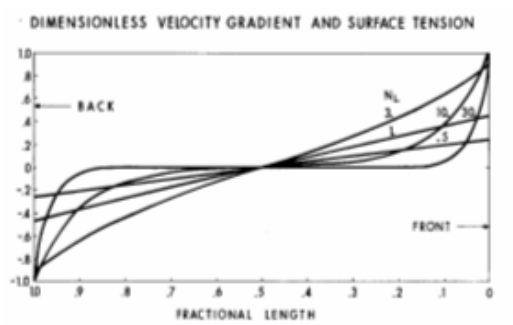


Equation:

$$\Delta p = 4.52\left(\frac{\sigma}{R}\right)\left(\frac{3\mu U}{\sigma}\right)^{2/3}$$

<u>Train of Bubbles or Foam in a Smooth Capillary.</u> The resistance to flow of a single bubble in a tube is the starting point for the flow of a coarse foam through a tube. The contributing factors to the pressure drop are the slugs of liquid, interface deformation as described by Bretherton, and the additional resistance due to surface tension gradients. (Hirasaki and Lawson, 1985). Assume that the tube is small enough that the bubble train consists of bubbles separated by individual lamella. In this case the radius of curvature of the meniscus is smaller that the radius of the capillary. However, it can be determined from the capillary radius, bubble size, and foam quality or gas fraction. The comparison of the apparent viscosity for the flow of foam through the capillary tube is shown in the attached figure. These results show that interfacial deformation alone greatly underestimate the resistance to flow. Analysis shows that surface tension gradients have a significant effect in retarding the motion of the interface of bubbles when surface active material is present. The following figure shows the dimensionless interfacial velocity for different values of the dimensionless surface tension gradient group. The final figure shows that satisfactory agreement with theory can be obtained if surface tension gradient effects are taken into account.





Assignment 11.2

Thickness of entrained film Compare the thickness of the film entrained by a plate pulled vertically out of a bath of liquid assuming one or the other of the two assumptions: (1) surface tension has no effect and (2) the capillary number is much less than unity. Make the thickness dimensionless with respect to a common reference length and compare the results as a function of the capillary number. You may accept the solution by Landau and Levich.

Numerical Simulation

The classical exact solutions of the Navier-Stokes equations are valuable for the insight they give and as an exact result that any approximate method should be able to duplicate to an acceptable precision. However, if we limit ourselves to exact solutions or even approximate perturbation or series solutions, a vast number of important engineering problems will be beyond reach. To meet this need, a number of computational fluid mechanics (CFM) codes have become commercially available. A student could use the CFM code as a virtual experiment to learn about fluid mechanics. While this may be satisfactory for someone who will become a design engineer, this is not sufficient for the researcher who may wish to solve a problem that has not been solved before. The user usually has to treat the CFM code as a "black box" and accept the result of the simulation as correct. The researcher must always be questioning if any computed result is valid and if not, what must be done to make it valid. Therefore I believe Ph.D. students should know how to develop their own numerical simulators rather than be just a user.

We will start with a working FORTAN and MATLAB code for solving very simple, generic problems in 2-D. The student should be able to examine each part of the code and understand everything with the exception of the algorithm for solving the linear system of equations. Then the student will change boundary conditions, include transient and nonlinear capabilities, include curvilinear coordinates, and compute pressure and stresses.

The Stream Function - Vorticity Method

Two-dimensional flow of an incompressible, Newtonian fluid can be formulated with the stream function and vorticity as dependent variables.
Equation:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -w$$
$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + v_x\frac{\partial w}{\partial x} + v_y\frac{\partial w}{\partial y} = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$
$$\text{where}$$
$$\mathbf{v} = (v_x,\ v_y,\ 0) = \nabla\times\mathbf{A} = \left(\frac{\partial\psi}{\partial y},\ -\frac{\partial\psi}{\partial x},\ 0\right), \quad \mathbf{A} = (0,0,\psi)$$
$$\mathbf{w} = (0,\ 0,\ w) = \left(0,\ 0,\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

We will begin developing a generic code by assuming steady, creeping (zero Reynolds number) flow with specified velocity on the boundaries in a rectangular domain. The PDE are now,
Equation:

$$\begin{aligned}\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} &= -w \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= 0\end{aligned} \quad , \qquad 0 < x < L_x, \quad 0 < y < L_y, \quad \mathrm{Re} = 0$$

This is a pair of linear, elliptic PDEs. The boundary conditions with specified velocity are
Equation:

$$
\begin{aligned}
&\mathbf{v}_{normal} = \text{specified} \\
&\mathbf{v}_{tangential} = \text{specified} = \quad O\left(U\right) \\
&\psi_{BC} = \int\limits_{boundary} \mathbf{v} \bullet \mathbf{n}\, ds \\
&w_{BC} = \nabla \times \mathbf{v}|_{boundary}
\end{aligned}
$$

The boundary conditions at a plane of symmetry are
Equation:

$$
\begin{aligned}
&\mathbf{n} \bullet \mathbf{v} = 0 \\
&n \bullet \nabla \mathbf{v} = 0 \\
&\psi = \text{constant} \\
&w = 0
\end{aligned}
$$

The PDE and definitions are made dimensionless with respect to reference quantities such that the variables are of the order of unity.
Equation:

$$
\begin{aligned}
&x^{*} = \frac{x}{L_x}, \quad y^{*} = \frac{y}{L_y}, \quad \alpha = \frac{L_x}{L_y} \\
&\mathbf{v}^{*} = \frac{\mathbf{v}}{U}, \quad \psi^{*} = \frac{\psi}{\psi_o}, \quad w^{*} = \frac{w}{w_o} \\
&\left[\frac{\psi_o}{L_x^2\, w_o}\right] \frac{\partial^2 \psi^{*}}{\partial x^{*2}} + \left[\frac{\psi_o}{L_y^2\, w_o}\right] \frac{\partial^2 \psi^{*}}{\partial y^{*2}} = -w^{*} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , \qquad 0 < x^{*} < 1, \quad 0 < y^{*} < 1 \\
&\qquad \frac{\partial^2 w^{*}}{\partial x^{*2}} + \alpha^2 \frac{\partial^2 w^{*}}{\partial y^{*2}} = 0 \\
&v_x{}^{*} = \left[\frac{\psi_o}{U\, L_y}\right] \frac{\partial \psi^{*}}{\partial y^{*}}, \quad v_y{}^{*} = -\left[\frac{\psi_o}{U\, L_x}\right] \frac{\partial \psi^{*}}{\partial x^{*}}
\end{aligned}
$$

We have four unspecified dimensionless groups and two unspecified reference quantities. We can specify two of the groups to equal unity and the other two are equal to the aspect ratio or its square.
Equation:

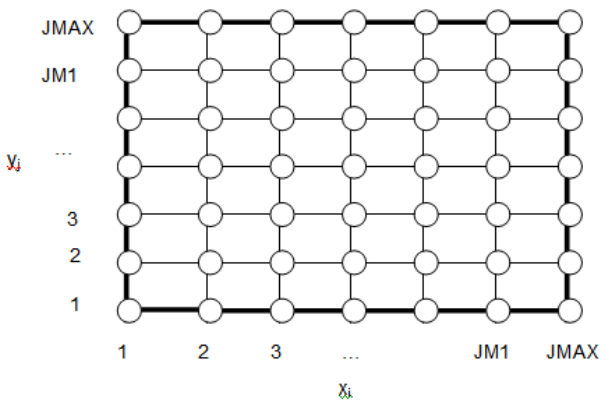
$$
\begin{aligned}
&\left[\frac{\psi_o}{L_x^2\, w_o}\right] = 1, \quad \left[\frac{\psi_o}{U\, L_x}\right] = 1 \\
&\Rightarrow \quad \psi_o = L_x\, U, \quad w_o = \frac{U}{L_x}
\end{aligned}
$$

The PDE and definitions with the * dropped are now as follows.
Equation:

$$
\begin{aligned}
&\frac{\partial^2 \psi}{\partial x^2} + \alpha^2 \frac{\partial^2 \psi}{\partial y^2} = -w \\
&\qquad\qquad\qquad\qquad\qquad\qquad , \qquad 0 < x < 1, \quad 0 < y < 1 \\
&\frac{\partial^2 w}{\partial x^2} + \alpha^2 \frac{\partial^2 w}{\partial y^2} = 0 \\
&v_x = \alpha \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x} \\
&w = \frac{\partial v_y}{\partial x} - \alpha \frac{\partial v_x}{\partial y}
\end{aligned}
$$

Finite Difference Approximation

The PDE and BC will be solved using a finite difference method. A grid point formulation rather than a grid block formulation will be used since the dependent variables are specified at the boundaries rather than their flux or normal derivative. The computation domain will be discretized such that the boundary conditions and dependent variables are evaluated at $x_1, x_2, \ldots, x_{JM1}, x_{JMAX}$ and $y_1, y_2, \ldots, y_{JM1}, y_{JMAX}$. The finite difference grid appears as follows.



Finite difference grid for discretizing PDE (grid point formulation)

The unit square is now discretized into *JMAX* by *JMAX* grid points where the boundary conditions and dependent variables will be evaluated. The grid spacing is $\delta = 1/(JMAX - 1)$. The first and last row and column are boundary values.

The partial derivatives will be approximated by finite differences. For example, the second derivative of vorticity is discretized by a Taylor's series.
Equation:

$$
\begin{aligned}
w_{i+1} &= w_i + \delta\left(\frac{\partial w}{\partial x}\right)_i + \frac{\delta^2}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)_i + \frac{\delta^3}{6}\left(\frac{\partial^3 w}{\partial x^3}\right)_i + \frac{\delta^4}{24}\left(\frac{\partial^4 w}{\partial x^4}\right)_{\overline{x}} \\
w_{i-1} &= w_i - \delta\left(\frac{\partial w}{\partial x}\right)_i + \frac{\delta^2}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)_i - \frac{\delta^3}{6}\left(\frac{\partial^3 w}{\partial x^3}\right)_i + \frac{\delta^4}{24}\left(\frac{\partial^4 w}{\partial x^4}\right)_{\underline{x}} \\
\left(\frac{\partial^2 w}{\partial x^2}\right)_i &= \frac{w_{i-1} - 2\,w_i + w_{i+1}}{\delta^2} - \frac{\delta^2}{24}\left[\left(\frac{\partial^4 w}{\partial x^4}\right)_{\overline{x}} + \left(\frac{\partial^4 w}{\partial x^4}\right)_{\underline{x}}\right] \\
&= \frac{w_{i-1} - 2\,w_i + w_{i+1}}{\delta^2} + O\left(\delta^2\right)
\end{aligned}
$$

The finite difference approximation to the PDE at the interior points results in the following set of equations.
Equation:

$$
\begin{aligned}
&\psi_{i+1,j} + \psi_{i-1,j} + \alpha^2\psi_{i,j+1} + \alpha^2\psi_{i,j-1} - 2\left(1+\alpha^2\right)\psi_{i,j} + \delta^2 w_{i,j} = 0 \\
&w_{i+1,j} + w_{i-1,j} + \alpha^2 w_{i,j+1} + \alpha^2 w_{i,j-1} \qquad\qquad - 2\left(1+\alpha^2\right)w_{i,j} = 0 \\
&i, j = 2,\ 3,\ \cdots JMAX - 1
\end{aligned}
$$

The vorticity at the boundary is discretized and expressed in terms of the components of velocity at the boundary, the stream function values on the boundary and a stream function value in the interior grid. (A greater accuracy is possible by using two interior points.) The stream function at the first interior point $(i = 2)$ from the x boundary is written with a Taylor's series as follows.
Equation:

$$\begin{aligned}
\psi_2 &= \psi_1 \pm \delta\left(\frac{\partial \psi}{\partial x}\right)_1 + \frac{\delta^2}{2}\left(\frac{\partial^2 \psi}{\partial x^2}\right)_1 + O\left(\delta^3\right) \\
\left(\frac{\partial^2 \psi}{\partial x^2}\right)_1 &= \frac{2(\psi_2 - \psi_1)}{\delta^2} \mp \frac{2}{\delta}\left(\frac{\partial \psi}{\partial x}\right)_1 + O(\delta) \\
&= \frac{2(\psi_2 - \psi_1)}{\delta^2} \pm \frac{2}{\delta} v_y^{BC} + O(\delta) \\
w_1^{BC} &\equiv \left(\frac{\partial v_y}{\partial x} - \alpha \frac{\partial v_x}{\partial y}\right)^{BC} \\
&= -\left(\frac{\partial^2 \psi}{\partial x^2}\right)_1 - \alpha \frac{\partial v_x^{BC}}{\partial y} \\
&= -\frac{2\left(\psi_2 - \psi_1^{BC}\right)}{\delta^2} \mp \frac{2}{\delta} v_y^{BC} - \alpha \frac{\partial v_x^{BC}}{\partial y} + O(\delta)
\end{aligned}$$

The choice of sign depends on whether x is increasing or decreasing at the boundary. Similarly, on a y boundary,
Equation:

$$w_1^{BC} = -\frac{2\alpha^2\left(\psi_2 - \psi_1^{BC}\right)}{\delta^2} \pm \frac{2\alpha\, v_x^{BC}}{\delta} + \frac{\partial v_y^{BC}}{\partial x} + O(\delta)$$

The boundary condition on the stream function is specified by the normal component of velocity at the boundaries. Since we have assumed zero normal component of velocity, the stream function is a constant on the boundary, which we specify to be zero.

The stream function at the boundary is calculated from the normal component of velocity by numerical integration using the trapezodial rule, e.g.,

$\psi_j = \psi_{j-1} + \frac{\delta}{2}\left[\left(v_x\right)_{j-1} + \left(v_x\right)_j\right]/\alpha$

Solution of Linear Equations

The finite difference equations for the PDE and the boundary conditions are a linear system of equations with two dependent variables. The dependent variables at a x_i, y_j grid point will be represented as a two component vector of dependent variables,
Equation:

$$\mathbf{u}_{i,j} = \begin{bmatrix} \psi_{i,j} \\ w_{i,j} \end{bmatrix}$$

The pair of equations for each grid point can be represented in the following form
Equation:

$$\begin{aligned}
&\mathbf{a}_{i,j}\,\mathbf{u}_{i+1,j} + \mathbf{b}_{i,j}\,\mathbf{u}_{i-1,j} + \mathbf{c}_{i,j}\,\mathbf{u}_{i,j+1} + \mathbf{d}_{i,j}\,\mathbf{u}_{i,j-1} + \mathbf{e}_{i,j}\,\mathbf{u}_{i,j} = \mathbf{f}_{i,j} \\
&i, j = 2,\ 3,\ \cdots JMAX - 1
\end{aligned}$$

Each coefficient is a 2x2 matrix. For example,
Equation:

$$\mathbf{e}_{i,j}\,\mathbf{u}_{i,j} = \begin{bmatrix} e_{i,j,1,1}\,\psi_{i,j} + e_{i,j,1,2}\,w_{i,j} \\ e_{i,j,2,1}\,\psi_{i,j} + e_{i,j,2,2}\,w_{i,j} \end{bmatrix}$$

The components of the 2x2 coefficient matrix are the coefficients from the difference equations. The first row is coefficients for the stream function equation and the second row is coefficients for the vorticity equation. The first column is coefficients for the stream function variable and the second column is the coefficients for the vorticity variable. For example, at interior points not affected by the boundary conditions,
Equation:

$$\begin{aligned}
\mathbf{a}_{i,j} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mathbf{b}_{i,j} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mathbf{c}_{i,j} &= \begin{bmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{bmatrix} \\
\mathbf{d}_{i,j} &= \begin{bmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{bmatrix} \\
\mathbf{e}_{i,j} &= \begin{bmatrix} -2\left(1+\alpha^2\right) & \delta^2 \\ 0 & -2\left(1+\alpha^2\right) \end{bmatrix} \\
\mathbf{f}_{i,j} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

The coefficients for the interior grid points adjacent to a boundary are modified as a result of substitution the boundary value of stream function or the linear equation for the boundary vorticity into the difference equations. The stream function equation is coupled to the vorticity with the $\mathbf{e}_{ij}$ coefficient and the vorticity equation is coupled to the stream function through the boundary conditions. For example, at a $x = 0$ boundary, the coefficients will be modified as follows.
Equation:

$$\begin{aligned}
& i = 2 \\
& f\left(i,j,1\right) = f\left(i,j,1\right) - b\left(i,j,1,1\right)^{*}\psi_{1,j}^{BC} \\
& f\left(i,j,2\right) = f\left(i,j,2\right) - b\left(i,j,2,2\right)^{*}\left(2^{*}\psi_{1,j}^{BC}/\delta^2 - 2^{*}v_v^{BC}\left(j\right)/\delta - \alpha\frac{\partial v_x^{BC}}{\partial y}\right) \\
& e\left(i,j,2,1\right) = e\left(i,j,2,1\right) - b\left(i,j,2,2\right)^{*}2/\delta^2
\end{aligned}$$

If the boundary condition is that of zero vorticity as on a surface of symmetry, it is necessary to only set to zero the coefficient that multiplies the boundary value of vorticity, i.e.,

$$\begin{aligned}
& i = 2 \\
& b(i,j,2,2) = 0, \quad \text{when } w(1,j) = 0
\end{aligned}$$

The linear system of equations and unknowns can be written in matrix form by ordering the equations with the i index changing the fastest, i.e., $i = 2, 3, \ldots, JMAX - 1$. The linear system of equations can now be expressed in matrix notation as follows.

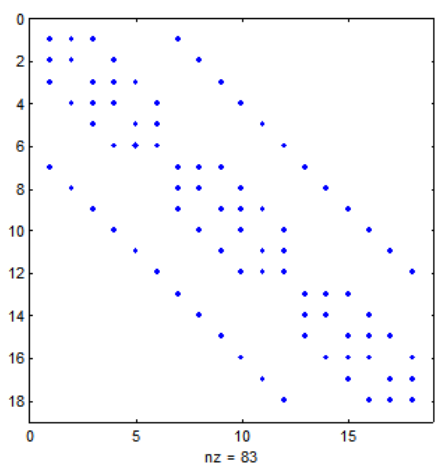
Equation:

$$\mathbf{A} \bullet \mathbf{x} = \mathbf{F}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{u}_{2,2} \\ \mathbf{u}_{3,2} \\ \vdots \\ \mathbf{u}_{JM1,JM1} \end{bmatrix} = \begin{bmatrix} \psi_{2,2} \\ w_{2,2} \\ \psi_{3,2} \\ w_{3,2} \\ \vdots \\ \psi_{JM1,JM1} \\ w_{JM1,JM1} \end{bmatrix}$$

The non-zero coefficients of coefficient matrix A for $JMAX = 5$ is illustrated below. Since the first and last grid points of each row and column of grid points are boundary conditions, the number of grid with pairs of unknowns is only 3x3. Only one grid point is isolated from boundaries. The i,j location of the coefficients become clearer if a box is drawn to enclose each $2x2$ coefficient.



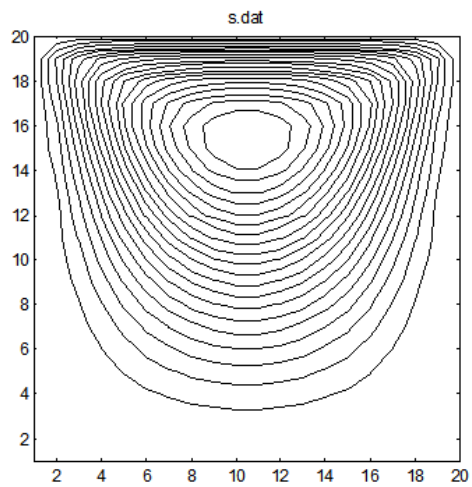
Most of the elements of the coefficient matrix are zero. In fact, the coefficient matrix is a block pentadiagonal sparse matrix and only the nonzero coefficients need to be stored and processed for solving the linear system of equations. The non-zero coefficients of the coefficient matrix are stored with the row-indexed sparse storage mode and the linear system of equations is solved by preconditioned biconjugate gradient method (Numerical Recipes, 1992).

The row-indexed sparse matrix mode requires storing the nonzero coefficients in the array, $sa(k)$, and the column number of the coefficient in the coefficient matrix in the array, $ija(k)$, where $k = 1, 2, \ldots$. The indices that are needed to identify the coefficients are the i, j grid location $(i, j = 2, 3, \ldots, JMAX - 1)$, the m equation and dependent variable identifier ($m = 1$ for stream function; $= 2$ for vorticity), and the IA row index of the coefficient matrix $(IA = 1, 2, 3, 4, \ldots, 2(JMAX - 2)2)$. The coefficient matrix has two

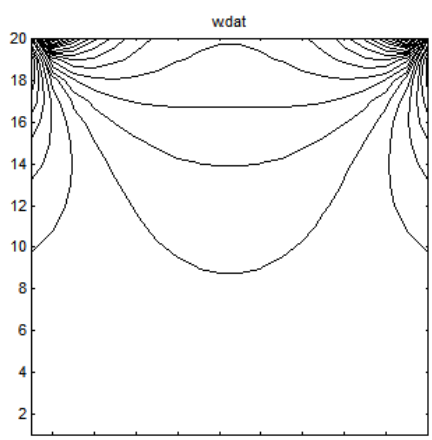
equations for each grid point, the first for the stream function and the second for vorticity. The total number of equations is $NN = 2(JMAX - 2)2$.

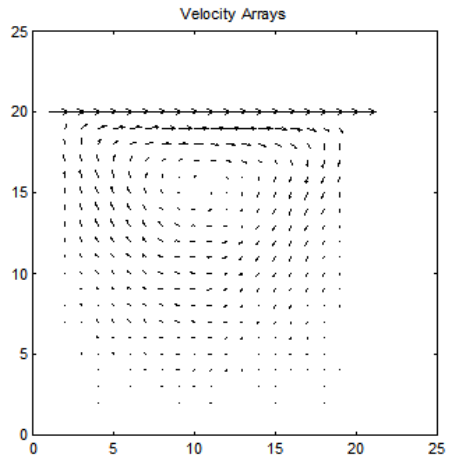
The diagonal elements of the coefficient array are first stored in the $sa(k)$ array. The first element of $ija(k)$, $ija(1) = NN + 2$ and can be used to determine the size of the coefficient matrix. The algorithm then cycles over the pairs of rows in the coefficient matrix while keeping track of the i,j locations of the grid point and cycling over equation $m = 1 \text{ and } 2$. The off-diagonal coefficients are stored in $sa(k)$ and the column number in the coefficient matrix stored in $ija(k)$. As each row is completed, the k index for the next row is stored in the first NN elements of ija.

A test problem for this code is a square box that has one side sliding as to impart a unit tangential component of velocity along this side. All other walls are stationary. The stream function, vorticity, and velocity contours for this problem with a 20x20 grid is illustrated below.



Stream function contours for square with top side sliding





Velocity vectors for square with top side sliding

Assignment 12.1
Use the code in the */numer/* directory of the CENG501 web site as a starting point. Add the capability to include arbitrarily specified normal component of velocity at the boundaries. The stream function at the boundary should be numerically calculated from the specified velocity at the boundary. Test the code with Couette and Plane-Poiseuille flows first in the x and then in the y directions. Display vector plot of velocity and contour plots of stream function and vorticity. Show your analysis and code.

Assignment 12.2
Form teams of two or three and do one of the following additions to the code. Find suitable problems to test code.

1. Calculate pressure and stress. Test problems: Couette and Plane-Poiseuille flows.
2. Calculate transient and finite Reynolds number flows. Test problems: Plate suddenly set in motion and flow in cavity.
3. Add option for cylindrical polar coordinates. Use the coordinate transformation, $z = \ln r$. Test problems: line source and annular rotational Couette flow.

Form new teams of two or three with one member that developed each part of the code. Include all features into the code. Test the combined features. Work together as a team to do the following simulation assignments.

Assignment 12.3
Boundary layer on flat plate. Compute boundary layer velocity profiles and drag coefficient and compare with the Blasius solution. What assumption is made about the value of the Reynolds in the Blasius solution?

Assignment 12.4
Flow around cylinder. Assume potential flow far from the cylinder. Calculate drag coefficient as a function of Reynolds number and compare with literature.

Assignment 12.5
Flow around cylinder. Find the Reynolds number at which boundary layer separation occurs.

FORTRAN on Owlnet

Modern FORTRAN compilers in the MSWindows are very user friendly. They have built in project workspace routines, documentation, and integrated editor and debugger. If you use the FORTRAN compiler on Owlnet, it is useful to know how to use *make* files to compile and link FORTRAN code. The following is an example *make* file. You may give it the name, *makefile*, with no extension.

```
f77 -c streamf1.f;
f77 -c bc1.f;
f77 -c coef1.f;
f77 -c asolve.f;
f77 -c atimes.f;
f77 -c dsprsax.f;
f77 -c dsprstx.f;
f77 -c linbcg.f;
f77 -c snrm.f;
f77 -
o exe streamf1.o bc1.o coef1.o asolve.o atimes.o dsprsax.o dsprstx.o linb
cg.o snrm.o
```

You have to give yourself permission to execute the make file with the one-time command,

```
chmod +x makefile
```

The line with the *-c* option compiles the source code and makes an object code file with the extension of .o. The line with the *-o* option links the object files into the executable file called *exe*. This executable file can be executed by typing its name, *exe*, on the command line.

This example *makefile* is not very efficient because it will compile every source code listed. There are instructions to recompile only the source code that has been modified or "touched" since the last time the *makefile* was invoked. However, I do not recall the instructions.

If you are not familiar with FORTRAN, you should get a paperback book on FORTRAN-77 or FORTRAN-90. An example is D. M. Etter, *Structured FORTRAN 77 for Engineers and Scientists*. The following is a tutorial from the CAAM 211 webpage. http://www.caam.rice.edu/ caam211/JZoss/project1.html

Computer facilities that have compilers usually have an interactive debugger. However, I have not found how to access the debugger on *owlnet*. The debugger with the Microsoft FORTRAN Powerstation (now HP FORTRAN) is very easy to use.

Transients and Finite Reynolds Number

The development of numerical simulators should start with a problem for which exact solutions exist so that the numerical algorithm can be verified. Steady, creeping flow has many exact solutions and thus it was used as the starting point for developing a numerical simulator for the Navier-Stokes equations. The student should verify that their code is able to duplicate exact solutions to any desired degree of accuracy.

The next stage in the development of the code is to add transients and finite Reynolds numbers. Finite Reynolds number results in nonlinear terms in the Navier-Stokes equation. Algorithms for solving linear systems of equations such as the conjugate gradient methods solves linear systems in one call to the routine. Nonlinear terms need to be iterated or lagged during the transient solution. Thus nonlinear steady state flow may be solved as the evolution of a transient solution from initial conditions to steady state.

The transient and nonlinear terms now need to be included when making the vorticity equation dimensionless.

Equation:

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + v_x \frac{\partial w}{\partial x} + v_y \frac{\partial w}{\partial y} = \frac{\mu}{\rho}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$
$$t^* = \frac{t}{t_o}$$
$$\frac{\partial w^*}{\partial t^*} + \left[\frac{U\,t_o}{L_x}\right] v_x^* \frac{\partial w^*}{\partial x^*} + \alpha\left[\frac{U\,t_o}{L_x}\right] v_y^* \frac{\partial w^*}{\partial y^*} = \left[\frac{\mu\,t_o}{\rho\,L_x^2}\right]\left[\frac{\partial^2 w^*}{\partial x^{*2}} + \alpha^2 \frac{\partial^2 w^*}{\partial y^{*2}}\right]$$

Two of the dimensionless groups can be set to unity by expressing the reference time in terms of the ratio of the characteristic velocity and characteristic length. The remaining dimensionless group is the Reynolds number.
Equation:

$$\left[\frac{U\,t_o}{L_x}\right] = 1 \quad \Rightarrow \; t_o = \frac{L_x}{U}$$
$$\left[\frac{\mu\,t_o}{\rho\,L_x^2}\right] = \left[\frac{\mu}{\rho\,U\,L_x}\right] = \frac{1}{\mathrm{Re}}$$

The dimensionless equation for vorticity with the * dropped is
Equation:

$$Re\left[\frac{\partial w}{\partial t} + v_x \frac{\partial w}{\partial x} + \alpha\, v_y \frac{\partial w}{\partial y}\right] = \frac{\partial^2 w}{\partial x^2} + \alpha^2 \frac{\partial^2 w}{\partial y^2}$$

The convective derivative can be written in conservative form by use of the continuity equation.
Equation:

$$(v_i\, w_j)_{,i} = v_{i,i}\, w_j + v_i\, w_{j,i}$$
$$\text{but}$$
$$v_{i,i} = 0, \quad \text{for incompressible flow}$$
$$\text{thus}$$
$$v_i\, w_{j,i} = (v_i\, w_j)_{,i}$$
$$\text{or}$$
$$\mathbf{v}\,\nabla\mathbf{w} = \nabla \bullet \mathbf{v}\,\mathbf{w}$$
$$Re\left[\frac{\partial w}{\partial t} + \frac{\partial v_x w}{\partial x} + \alpha\,\frac{\partial v_y w}{\partial y}\right] = \frac{\partial^2 w}{\partial x^2} + \alpha^2 \frac{\partial^2 w}{\partial y^2}$$

We will first illustrate how to compute the transient, linear problem before tackling the nonlinear terms. Stability of the transient finite difference equations is greatly improved if the spatial differences are evaluated at the new, $n+1$, time level while the time derivative is evaluated with a backward-difference method.
Equation:

$$\frac{\partial w}{\partial t} \approx \frac{w_{i,j}^{n+1} - w_{i,j}^{n}}{\Delta t} + O(\Delta t) = \frac{\Delta w_{i,j}}{\Delta t} + O(\Delta t)$$
$$\Delta w_{i,j} = w_{i,j}^{n+1} - w_{i,j}^{n}$$

Recall that the finite difference form of the steady state equations were expressed as a system of equations with coefficients, a, b, , ect. The vorticity equation for the i,j grid point will be rewritten but now with the transient term included. It is convenient to solve for Δw rather than w^{n+1} at each time step.

Equation:

$$a\,\Delta w_{i+1,j}^{n+1} + b\Delta\, w_{i-1,j}^{n+1} + c\,\Delta\, w_{i,j+1}^{n+1} + d\,\Delta w_{i,j-1}^{n+1} + e\,\Delta w_{i,j}^{n+1} - \frac{\delta^2 \mathrm{Re}}{\Delta t}\,\Delta w_{i,j}^{n+1}$$
$$= f - a\,w_{i+1,j}^{n} - b\,w_{i-1,j}^{n} - c\;w_{i,j+1}^{n} - d\,w_{i,j-1}^{n} - e\,w_{i,j}^{n}$$

We can see that this equation is of the same form as before with only the coefficients e and f of the vorticity equation modified.
Equation:

$$e' = e - \frac{\delta^2 Re}{\Delta t}$$
$$f' = f - a\,w_{i+1,j}^{n} - b\,w_{i-1,j}^{n} - c\;w_{i,j+1}^{n} - d\,w_{i,j-1}^{n} - e\,w_{i,j}^{n}$$

Since the variable in the linear system of equations is now Δw rather than w, the equations for the stream function also need to be modified.
Equation:

$$a_{1,1}\,\psi_{i+1,j} + b_{1,1}\,\psi_{i-1,j} + c_{1,1}\,\psi_{i,j+1} + d_{1,1}\,\psi_{i,j+1} + e_{1,1}\,\psi_{i,j} + e_{1,2}\,\Delta w_{i,j}$$
$$= f_{1,1} - e_{1,2}\,w_{i,j}^{n}$$
$$\text{thus}$$
$$f' = f_{1,1} - e_{1,2}\,w_{i,j}^{n}$$

These modifications to the coefficients should be made after the coefficients are updated for convection and boundary conditions.

The numerical calculations will start from some initial value of vorticity and proceed to steady state. The size of the time step is important if accuracy of the transient solution is of interest. The truncation error of the time finite difference expression is approximately the product $\Delta t\;\frac{\partial^2 w}{\partial t^2}$. The time step size can be chosen as to keep this value approximately constant. Thus the time step size will be chosen as to limit the maximum change in the magnitude of w over a time step. Let Δw_{max} be the maximum change in w over the previous time step and Δw_{spec} be the specified value of the desired maximum change in w. The new time step can be estimated from the following expression.
Equation:

$$\Delta t_{new} = \Delta t_{old}\frac{\Delta w_{spec}}{\Delta w_{\max}}$$

The initial time step size needs to be specified, and the new time step may be averaged with the old or constrained to increase by not greater than some factor.

Now that we have a means of calculating the transients solution, this approach can be used to treat the nonlinear terms. Recall that when the equations were linear, the system of equations had constant coefficients and the conjugate-gradient linear solver solved the linear system of equations.
Equation:

$$\mathbf{A}\,\mathbf{x} = \mathbf{F}$$

The convective derivatives have a product of velocity and vorticity, which can be expressed as a product of the derivative of stream function and vorticity. One approximation would be to use the value of the stream

function from the old time step to calculate the coefficients for the new time step. An alternative is use a predictor-corrector approach, similar to the Runge-Kutta solution for quasilinear ordinary differential equations. The "predictor" step estimates the solution at the new time step using the coefficients calculated from the stream function of the previous time step. This gives an estimate of the stream function at the new time step. This estimated stream function is then used to evaluate the coefficients for the "corrector" step. The steps in the calculations are as follows.
Equation:

$$\mathbf{A}\left(\mathbf{x}^{n}\right)\mathbf{x}^{n+1^{*}} = \mathbf{F}\left(\mathbf{x}^{n}\right)$$
$$\mathbf{A}\left(\mathbf{x}^{n+1^{*}}\right)\mathbf{x}^{n+1} = \mathbf{F}\left(\mathbf{x}^{n+1^{*}}\right)$$

The choice of the finite difference expression for the convective derivative is very important for the stability of the solution. We will illustrate here treatment of only the x derivative. Suppose $i-1$, i, and $i+1$ are the grid points where the stream function and vorticity are evaluated and $i-1/2$ and $i+1/2$ are the mid-points where the velocity-vorticity products are evaluated.
Equation:

$$\frac{\partial u\,w}{\partial x} \approx \frac{(u\,w)_{i+1/2} - (u\,w)_{i-1/2}}{\Delta x}$$

The dependent variables are known only at the grid points and interpolation will be needed to evaluate in between location. If the product is evaluated at $i-1/2$ and $i+1/2$ by using the arithmetic average of the values at the grid points on either side, the numerical solution will tend to oscillate if the convective transport dominates the diffusive (viscous) transport. It is preferable to determine the upstream direction from the sign of $u_{i+1/2}$ or $u_{i-1/2}$ and use the vorticity from the upstream grid point. This is illustrated below.

```
    ui-1/2 = 0.5*(u(i,j)+u(i-1,j))
        if ui-1/2 >0 then
                (uw)i-1/2 = ui-1/2 wi-1 ;
b'=b + $\delta$ Re ui-1/2
        else
                (uw)i-1/2 = ui-1/2 wi ;         e'=e + $\delta$Re ui-1/2
        endif

        ui+1/2 = 0.5*(u(i,j)+u(i+1,j))
        if uI+1/2 >0 then
                (uw)i+1/2 = ui+1/2 wi ;
e'=e - $\delta$  Re ui+1/2
        else
                (uw)i+1/2 = ui+1/2 wi+1 ;
a'=a - $\delta$  Re ui+1/2
        endif
```

The y-direction will be similar except the aspect ratio, , must be included. These modifications to the coefficients must be made before the coefficients are updated for the boundary conditions.

Calculation of Pressure

The vorticity-stream function method does not require calculation of pressure to determine the flow field. However, if the force or drag on a body or a conduit is of interest, the pressure must be computed to determine the stress field. Here we will derive the Poisson equation for pressure and determine the boundary

conditions using the equations of motion. If the pressure is desired only at the boundary, it may be possible to integrate the pressure gradient determined from the equations of motion. The dimensionless equations of motion for incompressible flow of a Newtonian fluid are as follows.
Equation:

$$\begin{aligned}
\frac{d\mathbf{v}^*}{dt^*} &= \frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{v}^* \bullet \nabla^* \mathbf{v}^* = \frac{\partial \mathbf{v}^*}{\partial t^*} + \nabla^* \bullet (\mathbf{v}^* \mathbf{v}^*) = \\
& -\nabla^* P^* + \frac{1}{Re} \nabla^{*2} \mathbf{v}^* \\
& \text{where} \\
\mathbf{v}^* &= \frac{\mathbf{v}}{U}, \quad \mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad t^* = \frac{U}{L} t, \quad P^* = \frac{P}{\rho U^2}, \quad \nabla^* = L \nabla \\
P &= p - \rho g z, \quad Re = \frac{\rho U L}{\mu}
\end{aligned}$$

Here all coordinates were scaled with respect to the same length scale. The aspect ratio must be included in the final equation if the coordinates are scaled with respect to different length scales. We will drop the * and use x and y for the coordinates and u and v as the components of velocity.

The following derivation follows that of Hoffmann and Chiang (1993). The equations of motion in 2-D in conservative form is as follows.
Equation:

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (u\,v)}{\partial y} &= -\frac{\partial P}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
\frac{\partial v}{\partial t} + \frac{\partial (u\,v)}{\partial x} + \frac{\partial (v^2)}{\partial y} &= -\frac{\partial P}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)
\end{aligned}$$

The Laplacian of pressure is determined by taking the divergence of the equations of motion. We will carry out the derivation step wise by first taking the x-derivative of the x-component of the equations of motion.
Equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 P}{\partial x^2} + \frac{1}{Re}\frac{\partial}{\partial x}\left(\nabla^2 u\right)$$

Two pair of terms cancels because of the equation of continuity for incompressible flow.
Equation:

$$\begin{aligned}
& \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \\
& u\frac{\partial^2 v}{\partial x \partial y} + u\frac{\partial^2 u}{\partial x^2} = u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \\
& \text{Thus} \\
& \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 P}{\partial x^2} + \frac{1}{\mathrm{Re}}\frac{\partial}{\partial x}\left(\nabla^2 u\right)
\end{aligned}$$

Similarly, the y-component of the equations of motion become
Equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 P}{\partial y^2} + \frac{1}{Re}\frac{\partial}{\partial y}\left(\nabla^2 v\right)$$

The x and y-components of the above equations are now added together and several pairs of terms cancels pair-wise.
Equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$
$$\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$
$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

Therefore the equations reduce to
Equation:

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=-\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)$$

The left-hand side can be further reduced by consideration of the continuity equation as follows.
Equation:

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}=0$$
$$\text{from which}$$
$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2=-2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}$$

Upon substitution, we have
Equation:

$$-\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)=2\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$$

This equation can be written in terms of the stream function.
Equation:

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}=2\left[\frac{\partial^2 \psi}{\partial x^2}\ \frac{\partial^2 \psi}{\partial y^2}-\left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2\right]$$

This is the equation when the coordinates have the same reference length. If the coordinates are normalized with respect to different lengths, then the equation is as follows.
Equation:

$$\frac{\partial^2 P}{\partial x^2}+\alpha^2\frac{\partial^2 P}{\partial y^2}=2\alpha^2\left[\frac{\partial^2 \psi}{\partial x^2}\ \frac{\partial^2 \psi}{\partial y^2}-\left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2\right]$$

The Poisson equation for pressure needs to have boundary conditions for solution. The equations of motion give an expression for the pressure gradient. The normal derivative of pressure at the boundary can be determined for the Neumann-type boundary condition. The equations of motion usually simplify at boundaries if the boundary has no slip BC or if it has parallel flow in the direction either parallel or normal to the boundary. For example, for no-slip

Equation:

$$\begin{aligned} \frac{\partial P}{\partial x} &= \frac{\alpha^2}{Re}\frac{\partial^2 u}{\partial y^2}, \quad \text{at } y=c,\ u=v=0 \\ \alpha\frac{\partial P}{\partial y} &= \frac{1}{Re}\frac{\partial^2 v}{\partial x^2}, \quad \text{at } x=c,\ u=v=0 \end{aligned}$$

If the flow is parallel and in the direction normal to the boundary

Equation:

$$\begin{aligned} \frac{\partial P}{\partial x} &= \frac{\alpha^2}{Re}\frac{\partial^2 u}{\partial y^2}, \quad y=c,\ v=0,\ u=u\left(y\right) \\ \alpha\frac{\partial P}{\partial y} &= \frac{1}{Re}\frac{\partial^2 v}{\partial x^2}, \quad x=c,\ u=0,\ v=v\left(x\right) \end{aligned}$$

Cylindrical-Polar Coordinates

The code for the numerical solution of the Navier-Stokes equations in Cartesian coordinates can be easily modified to cylindrical-polar coordinates. The coordinate transformation is first illustrated for the Laplacian operator.

Equation:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}, \qquad r_1 \le r \le r_2, \quad 0 \le \theta \le \theta_o$$

The independent variables are made dimensionless with respect to the boundary parameters.

Equation:

$$r^* = \frac{r}{r_1}, \quad \theta^* = \frac{\theta}{\theta_o}, \qquad 1 \le r^* \le \frac{r_2}{r_1} = \beta, \quad 0 \le \theta^* \le 1$$

The Lapacian operator with the dimensionless coordinates after dropping the * is now,

Equation:

$$\nabla^2\psi = \frac{1}{r_1^2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{\theta_o^2}\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right], \qquad 1 \le r \le \beta, \quad 0 \le \theta \le 1$$

The radial coordinate is transformed to the logarithm of the radius.

Equation:

$$\begin{aligned} & z = \frac{\ln r}{\ln\beta} = \gamma \ln r, \qquad 0 \le z \le 1, \quad \gamma = \frac{1}{\ln(r_2/r_1)} \\ & r = \exp\left(z/\gamma\right) \\ & \frac{\partial}{\partial r} = \frac{dz}{dr}\frac{\partial}{\partial z} = \frac{\gamma}{r}\frac{\partial}{\partial z} \\ & \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \frac{\gamma^2}{r^2}\frac{\partial^2\psi}{\partial z^2} \end{aligned}$$

The Laplacian operator is now as follows,
Equation:

$$\nabla^2\psi = \frac{1}{r_1^2}\left[\frac{\gamma^2}{r^2}\frac{\partial^2\psi}{\partial z^2} + \frac{\alpha^2}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right], \quad 0 \le z \le 1, \quad 0 \le \theta \le 1, \quad \alpha = \frac{1}{\theta_o}$$

The finite difference expression for the Laplacian will be same as that for Cartesian coordinates with (z, θ) substituted for (x, y) except for γ^2 and r^2 factors.
Equation:

$$\begin{aligned}
&0 \le z_i \le 1, \quad 0 \le \theta_j \le 1, \quad i, j = 1, 2, ..., JMAX \\
&\delta = \tfrac{1}{JMAX-1} \\
&z_i = \delta(i-1), \quad \theta_j = \delta(j-1)
\end{aligned}$$

The curl operator is modified from that in Cartesian coordinates.
Equation:

$$\begin{aligned}
v_r &= \tfrac{\alpha}{r}\tfrac{\partial\psi}{\partial\theta} \\
v_\theta &= -\tfrac{\partial\psi}{\partial r} = -\tfrac{\gamma}{r}\tfrac{\partial\psi}{\partial z} \\
w &= \tfrac{1}{r}\tfrac{\partial(r\,v_\theta)}{\partial r} - \tfrac{\alpha}{r}\tfrac{\partial v_r}{\partial\theta} \\
&= \tfrac{\gamma}{r^2}\tfrac{\partial(r\,v_\theta)}{\partial z} - \tfrac{\alpha}{r}\tfrac{\partial v_r}{\partial\theta} \\
&= -\tfrac{\gamma^2}{r^2}\tfrac{\partial^2\psi}{\partial z^2} - \tfrac{\alpha^2}{r^2}\tfrac{\partial^2\psi}{\partial\theta^2}
\end{aligned}$$

The vorticity boundary conditions with the transformed coordinates is for the z boundary,
Equation:

$$w_1^{BC} = -\frac{\gamma^2}{r^2}\frac{2}{\delta^2}\left(\psi_2 - \psi_1^{BC}\right) \mp \frac{\gamma}{r}\frac{2}{\delta}v_\theta^{BC} - \frac{\alpha}{r}\frac{\partial v_r^{BC}}{\partial\theta}$$

and for the θ boundary,
Equation:

$$w_1^{BC} = -\frac{\alpha^2}{r^2}\frac{2}{\delta^2}\left(\psi_2 - \psi_1^{BC}\right) \pm \frac{\alpha}{r}\frac{2}{\delta}v_r^{BC} + \frac{\gamma}{r^2}\frac{\partial\left(r v_\theta^{BC}\right)}{\partial z}$$

The stream function at the boundaries are expressed different from that in Cartesian coordinates.
Equation:

$$\begin{aligned}
&d\psi = (r\,v_r/\alpha)\,d\theta, \quad \text{at } z \text{ boundary} \\
&d\psi = -(r\,v_\theta/\gamma)\,dz, \quad \text{at } \theta \text{ boundary}
\end{aligned}$$

The convective terms are expressed different from that in Cartesian coordinates.
Equation:

$$
\begin{aligned}
\frac{\partial(u\,w)}{\partial x} &\Rightarrow \frac{1}{r}\frac{\partial(r\,u\,w)}{\partial r} \\
&= \frac{\gamma}{r^2}\frac{\partial(r\,u\,w)}{\partial z} \\
&\approx \frac{\gamma}{r^2\,\delta}\Big[(r\,u\,w)_{i+1/2} - (r\,u\,w)_{i-1/2}\Big] \\
\alpha\frac{\partial(v\,w)}{\partial y} &\Rightarrow \frac{\alpha}{r}\frac{\partial(v\,w)}{\partial \theta} \\
&\approx \frac{\alpha}{r\,\delta}\Big[(v\,w)_{j+1/2} - (v\,w)_{j-1/2}\Big]
\end{aligned}
$$

The code should be written so one can have either Cartesian coordinates or transformed cylindrical-polar coordinates. A parameter will be needed to identify the choice of coordinates, e.g. $icase = 1$ for Cartesian coordinates and $icase = 2$ for transformed cylindrical-polar coordinates. Also, another parameter should be specified to identify the choice of boundary conditions, e.g., $ibc = 1$ for radial flow, $ibc = 2$ for Couette flow, and $ibc = 3$ for flow around a cylinder. Test cases with known solutions should be used to verify the code. The first case is radial, potential flow from a line source and the second is Couette flow in the annular region between two cylindrical surfaces.

Equation:

$$
\left.\begin{aligned} v_r &= 1/r \\ v_\theta &= 0 \end{aligned}\right\} \quad \text{radial flow}
$$

$$
\left.\begin{aligned} v_r &= 0 \\ v_\theta &= (r - 1/r)/(\beta - 1/\beta) \end{aligned}\right\} \quad \text{Couette flow}
$$

Flow around a cylinder needs a boundary condition for the flow far away from the cylinder. The flow very far away may be uniform translation. However, this condition may be so far away that it may result in loss of resolution near the cylinder. Another boundary condition that may be specified beyond the region of influence of the boundary layer is to use the potential flow past a cylinder. This boundary condition will not be correct in the wake of cylinder where the flow is disturbed by the convected boundary layer buts its influence may be minimized if the outer boundary is far enough.

Equation:

$$
\left.\begin{aligned} v_r &= \left(1 - 1/r^2\right) \quad \cos\theta \\ v_\theta &= -\left(1 + 1/r^2\right) \quad \sin\theta \end{aligned}\right\} \quad \text{Potential flow past cylinder}
$$

Potential flow will not be a valid approximation along the θ boundaries close to the cylinder. Here, one may assume a surface of symmetry, at least for the upstream side.